

# ON THE DIVERIO-TRAPANI CONJECTURE

Ya Deng\*

Institut Fourier, Université Grenoble Alpes

## Abstract

The aim of this work is to study the conjecture on the ampleness of Demailly-Semple bundles raised by Diverio and Trapani, and also obtain some effective estimates related to this problem.

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## 1 INTRODUCTION

In recent years, an important technique in studying hyperbolicity-related problems is *invariant* jet differentials  $E_{k,m}T_X^*$  introduced by J.-P. Demailly, which can be seen as a generalization to higher order of symmetric differentials, but invariant under the reparametrization. To prove hyperbolicity-type statements for projective manifolds, one needs to construct (many) global jet differentials vanishing on an ample divisor on the given manifold  $X$  (cf. Theorem 2.3 below). If one deal the with positivity for jet bundles of the complete intersection of hypersurfaces in  $\mathbb{P}^N$ , as was proved in [Div08], one cannot expect to achieve this for lower order jet differentials if the codimension of subvariety is small:

**Theorem 1.1.** (*Diverio*) *Let  $X \subset \mathbb{P}^N$  be a smooth complete intersection of hypersurfaces of any degree in  $\mathbb{P}^N$ . Then*

$$H^0(X, E_{k,m}^{GG}T_X^*) = 0$$

*for all  $m \geq 1$  and  $1 \leq k < \dim(X)/\text{codim}(X)$ .*

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\*Email address: Ya.Deng@univ-grenoble-alpes.fr

On the other hand, in principle, the positivity (or hyperbolicity) of a generic complete intersection in the projective space should be increased by cutting more and more with projective hypersurfaces of high degree. In [Deb05], Debarre verified this in the case of *abelian variety*, in which he proved that the intersection of at least  $\frac{N}{2}$  sufficiently ample general hypersurfaces in an  $N$ -dimensional abelian variety has ample cotangent bundle. Motivated by this result, he conjectured that the analogous statement holds in the projective space:

**Conjecture 1.1.** (Debarre) *The cotangent bundle of the intersection in  $\mathbb{P}^N$  of at least  $\frac{N}{2}$  general hypersurfaces of sufficiently high degree is ample.*

The first important result in this direction was obtained by Brotbek in [Bro14], where he was able to prove the Debarre conjecture for complete intersection surfaces in  $\mathbb{P}^4$ . Later, in [Bro15] he proved the ampleness of the cotangent bundle of the intersection of at least  $\frac{3n-2}{4}$  general hypersurfaces of high degree in  $\mathbb{P}^n$ . Very recently, based on the ideas and explicit methods arising in [Bro15], Brotbek and Darondeau [BD15] and independently S.-Y. Xie [Xie15, Xie16] proved the Debarre conjecture:

**Theorem 1.2.** (Brotbek-Darondeau, Xie) *Let  $X$  be any smooth projective variety of dimension  $N$ , and  $A$  a very ample line bundle on  $X$ , there exists a positive number  $d_N$  depending only on the dimension  $N$ , such that for each  $c \geq \frac{N}{2}$ , the complete intersection of  $c$  general hypersurfaces in  $|A^\delta|$  has ample cotangent bundle.*

Moreover, Xie was able to give an effective lower bound on hypersurface degrees  $d_N := N^{N^2}$ . Although the work by Brotbek and Darondeau is not effective on the lower bound  $d_N$ , growing from some interpretation of the cohomological computations in [Bro15], they established an elegant geometric construction, which defines a map  $\Psi$  from the projectivized relative cotangent bundle  $\mathbb{P}(\Omega_{X/S})$  to a certain family  $\mathcal{Y} \rightarrow \mathbf{G}$ , which we called *the universal Grassmannian* in Section 4, to construct a lot of global symmetric differential forms with a negative twist by pulling-back the positivity on  $\mathcal{Y}$ . In order to make the base locus empty, they applied the Nakamaye Theorem, which asserts that for a big and nef line bundle  $L$  on a projective variety, the augmented base locus  $\mathbb{B}_+(L)$  coincides with the null locus  $\text{Null}(L)$ , to the tautological line bundle  $\mathcal{L}$  on the universal Grassmannian  $\mathcal{Y}$ . In Section 4, we obtain an *effective* result (see Theorem 4.3) related to the Nakamaye Theorem they used, which is a bit weaker but still valid in their proof. Thus based on their work we can obtain a better lower bound (see also [Den16])

$$d_N = 4c_0(2N - 1)^{2c_0+1} + 6N - 3,$$

where  $c_0 := \lfloor \frac{N+1}{2} \rfloor$ .

On the other hand, by introducing a new compactification of the set of regular jets  $J_k T_X^{\text{reg}}/\mathbb{G}_k$ , Brotbek was able to fully develop the ideas in [BD15] to prove the Kobayashi conjecture [Bro16]. His statement is thus the following:

**Theorem 1.3.** (Brotbek) *Let  $X$  be a smooth projective variety of dimension  $n$ . For any very ample line bundle  $A$  on  $X$  and any  $d \geq d_{K,n}$ , a general hypersurface in  $|A^d|$  is Kobayashi hyperbolic. Here  $d_{K,n}$  depends only the dimension  $n$ .*

In [Bro16], the main new tool he constructed is the *Wronskians* on the Demailly-Semple tower, which associates sections of the line bundle to global invariant jet differentials. As there are certain insuperable obstructions to the positivity of the tautological line bundle on the Demailly-Semple towers, due to the compactification of the jet bundles (ref. [Dem95]), Brotbek introduced a clever way to blow-up the ideal sheaves defined by the Wronskians, which behaves well in families, so that he was able to apply the openness property of ampleness for the higher order jet bundles to prove the hyperbolicity for general hypersurfaces. In order to make the lower bound  $d_{K,n}$  in Theorem 1.3 effective, one needs to obtain some effective estimates arising in some noetherianity arguments. As well as the Nakamaye Theorem, there is another constant  $m_\infty(X_k, L)$  (see Section 2.3) which reflects the stability of Wronskian ideal sheaf when the positivity of the line bundle  $L$  increases. In Section 2.3 we study Brotbek's Wronskians and prove the *effective finite generation* for Wronskian ideal sheaf (Theorem 2.4), and thus based on Brotbek's result we were able to obtain an effective bound for the Kobayashi conjecture (see also [Den16])

$$d_{K,n} = n^{n+1}(n+1)^{2n+5}.$$

**Remark 1.1.** By using Siu's technique of slanted vector fields on higher jet spaces outlined in his survey [Siu02], and the Algebraic Morse Inequality by Demailly and Trapani, the first effective lower bound for the degree of the general hypersurface which is weakly hyperbolic (say that a variety  $X$  is weakly hyperbolic if all its entire curves lie in a proper subvariety  $Y \subsetneq X$ ) was given by Diverio, Merker and Rousseau [DMR10], where they confirmed the Green-Griffiths-Lang conjecture for generic hypersurfaces in  $\mathbb{P}^n$  of degree  $d \geq 2^{(n-1)^5}$ . Later on, by means of a very elegant combination of his holomorphic Morse inequalities and a probabilistic interpretation of higher order jets, Demailly was able to improve the lower bound to  $d \geq \left\lfloor \frac{n^4}{3} \left( n \log(n \log(24n)) \right)^n \right\rfloor$  [Dem10]. The latest best known bound was  $d \geq (5n)^2 n^n$  by Darondeau [Dar15]. In the recent published paper [Siu15], Siu provided more details to his strategy in [Siu02] to complete his proof of the Kobayashi conjecture, and the bound on the degree following [Siu15] are very difficult to make explicit.

In the same vein as the Debarre conjecture, in [DT10], Simone Diverio and Stefano Trapani raised the following generalized conjecture:

**Conjecture 1.2.** (Diverio-Trapani) Let  $X \subset \mathbb{P}^N$  be the complete intersection of  $c$  general hypersurfaces of sufficiently high degree. Then,  $E_{k,m}T_X^*$  is ample provided that  $k \geq \frac{N}{c} - 1$ , and therefore  $X$  is hyperbolic.

In this paper, based mainly on the elegant geometric methods in [BD15] and [Bro16] on the Debarre and Kobayashi conjectures, we prove the following theorem:

**Theorem A.** Let  $X$  be a projective manifold of dimension  $n$  endowed with a very ample line bundle  $A$ . Let  $Z \subset X$  be the complete intersection of  $c$  general hypersurfaces in  $|H^0(X, \mathcal{O}_X(dA))|$ . Then for any positive integer  $k \geq \frac{n}{c} - 1$ ,  $Z$  has almost  $k$ -jet ampleness (see Definition 2.1 below) provided that  $d \geq 2c(\lceil \frac{n}{c} \rceil)^{n+c+2} n^{n+c}$ . In particular,  $Z$  is Kobayashi hyperbolic.

Since our definition for almost 1-jet ampleness coincides with ampleness of cotangent bundle, then our Main Theorem integrates both the Kobayashi ( $c = 1$ ) and Debarre conjectures ( $c \geq \frac{n}{2}$ ), with some (non-optimal) effective estimates.

At the expense of a slightly larger bound, based on a factorization trick due to Xie [Xie15], we are able to prove the following stronger result:

**Theorem B.** Let  $X$  be a projective manifold of dimension  $n$  and  $A$  a very ample line bundle on  $X$ . For any  $c$ -tuple  $\mathbf{d} := (d_1, \dots, d_c)$  such that  $d_p \geq c^2 n^{2n+2c} (\lceil \frac{n}{c} \rceil)^{2n+2c+4}$  for each  $1 \leq p \leq c$ , for general hypersurfaces  $H_p \in |A^{d_p}|$ , their complete intersection  $Z := H_1 \cap \dots \cap H_c$  has almost  $k$ -jet ampleness provided that  $k \geq k_0$ .

Moreover, there exists a uniform  $(e_1, \dots, e_k) \in \mathbb{N}^k$  which only depends on  $n$ , such that  $\mathcal{O}_{Z_k}(e_1, \dots, e_k)$  is big and its augmented base locus

$$\mathbb{B}_+(\mathcal{O}_{Z_k}(e_1, \dots, e_k)) \subset Z_k^{\text{Sing}}$$

where  $Z_k^{\text{Sing}}$  is the set of points in  $Z_k$  which can not be reached by the  $k$ -th lift  $f_{[k]}(0)$  of any regular germ of curves  $f : (\mathbb{C}, 0) \rightarrow Z$ .

From the relation between tautological bundles on the Demailly-Semple towers and invariant jet bundles, we prove the following theorem on the Diverio-Trapani conjecture:

**Theorem C.** Set  $q := \mathcal{Z}_{\mathbf{d}} \rightarrow \prod_{p=1}^c |A^{d_p}|$  to be the universal family of  $c$ -complete intesections of hypersurfaces in  $\prod_{p=1}^c |A^{d_p}|$ , where  $d_p \geq c^2 n^{2n+2c} (\lceil \frac{n}{c} \rceil)^{2n+2c+4}$  for each  $1 \leq p \leq c$ . Set  $U \subset \prod_{p=1}^c |A^{d_p}|$  to be a Zariski open set of  $\prod_{p=1}^c |A^{d_p}|$  such that when restricted to  $\mathcal{X} := q^{-1}(U)$ ,  $q$  is a smooth fibration. Then for every  $j \gg 0$ , there exists a subbundle  $V_j \subset E_{k,jm}T_{\mathcal{X}/U}^*$  defined on  $\mathcal{X}$ , whose restriction to the general fiber  $Z$  of  $q$  is an ample vector bundle. Moreover, fix any  $x \in Z$ , and any regular  $k$ -jet of holomorphic curve  $[f] : (\mathbb{C}, 0) \rightarrow (Z, x)$ , then for every  $j \gg 0$  there exists global jet differentials  $P_j \in H^0(Z, V_j|_Z \otimes A^{-1})$  (hence they are of order  $k$  and weighted degree  $jm$ ) does not vanish when evaluated on the  $k$ -jet defined by  $(f', f'', \dots, f^{(k)})$ .

In other words, this theorem shows that, we can find a subbundle of the invariant jet bundle, which is ample, and its Demailly-Semple locus defined in [DR13, Section 2.1] is empty.

This paper is organized as follows. In Section 2, we will collect some definitions and notations, and recall some related results on the invariant jet differentials and their relations with the Demailly-Semple tower. In particular, in Section 2.3 we study the main tool initiated by Brotbek very recently in his proof of the Kobayashi conjecture (cf. [Bro16]), namely the Wronskians on the Demailly-Semple towers. Using the bundle of  $k$ -jet sections of a line bundle  $L$  as the intermediate stage between global sections of  $L$  and the Wronskians, we can obtain an “*effective finite generation*” of the  $k$ -th Wronskian ideal sheaf  $\mathfrak{w}(X_k, L)$  (see Theorem 2.4 below), which is essential in the effective estimate for the lower bound of the degree in the Main Theorems. Next, in Section 3, we provide the proofs of the Main Theorems. The general idea is a combination of the methods arising in [BD15] and [Bro16]: we reduce the general statement to the construction of an example with jet ampleness, and then apply the openness property to prove the jet ampleness for general complete intersections. Such an example arises from the complete intersections of families of hypersurfaces which are deformations of Fermat type hypersurfaces. Using Brotbek’s Wronskians we are able to construct a (rational) map  $\Psi$  from the Demailly-Semple towers of such complete intersections to the (more general) universal Grassmannian  $\mathcal{Y}$  (one need to blow-up the Wronskian ideal sheaf to resolve the indeterminacy of that rational map  $\Psi$ ), so that we can construct a lot of invariant jet differentials with a negative twist by pulling-back the positivity on  $\mathcal{Y}$ . To control the base locus of the jet bundle of the general complete intersection, the image of  $\Psi$  should avoid the locus of positive dimension fibers  $\mathcal{Y} \rightarrow \mathbf{G}$ . In Section 4, we study the augmented base locus of the tautological line bundle on the universal Grassmannians, where we give an explicit construction for a lot of divisors in certain linear systems to show that, their base locus avoid the non-finite loci  $G^\infty$ , to obtain an “almost” Nakamaye Theorem. This paper contains some results already appeared in [Den16]. Here our main theorems are more general and we provide more details to make the statements and proofs self-contained, which could supercede the paper [Den16].

## 2 TECHNICAL PRELIMINARIES AND LEMMAS

### 2.1 INVARIANT JET DIFFERENTIALS

Let  $(X, V)$  be a directed manifold, i.e. a pair where  $X$  is a complex manifold and  $V \subset T_X$  a holomorphic subbundle of the tangent bundle. One defines  $J_k V \rightarrow X$  to be the bundle of  $k$ -jets of germs of parametrized curves in  $X$ , that is, the set of equivalent classes of holomorphic maps  $f : (\mathbb{C}, 0) \rightarrow (X, x)$  which are tangent to  $V$ , with the equivalence relation  $f \sim g$  if and only if all derivatives  $f^{(j)}(0) = g^{(j)}(0)$  coincide for  $0 \leq j \leq k$ , when computed in some local coordinate system of  $X$  near  $x$ . From now on, if not specially mentioned, we always assume that  $V = T_X$ . The projection map  $p_k : J_k T_X \rightarrow X$  is simply taken to be  $[f] \mapsto f(0)$ . If  $(z_1, \dots, z_n)$  are local holomorphic coordinates on an open set  $\Omega \subset X$ , the elements  $[f]$  of any fiber  $J_{k,x}$ ,  $x \in \Omega$ , can be seen as  $\mathbb{C}^n$ -valued maps

$$f = (f_1, \dots, f_n) : (\mathbb{C}, 0) \rightarrow \Omega \subset \mathbb{C}^n,$$

and they are completely determined by their Taylor expansion of order  $k$  at  $t = 0$ :

$$f(t) = x + t f'(0) + \frac{t^2}{2!} f''(0) + \dots + \frac{t^k}{k!} f^{(k)}(0) + O(t^{k+1}).$$

In these coordinates, the fiber  $J_{k,x}$  can thus be identified with the set of  $k$ -tuples of vectors

$$(\xi_1, \dots, \xi_k) = (f'(0), f''(0), \dots, f^{(k)}(0)) \in \mathbb{C}^n.$$

Let  $\mathbb{G}_k$  be the group of germs of  $k$ -jets of biholomorphisms of  $(\mathbb{C}, 0)$ , that is, the group of germs of biholomorphic maps

$$t \mapsto \varphi(t) = a_1 t + a_2 t^2 + \dots + a_k t^k, \quad a_1 \in \mathbb{C}^*, a_j \in \mathbb{C}, j > 2,$$

in which the composition law is taken modulo terms  $t_j$  of degree  $j > k$ . Then  $\mathbb{G}_k$  is a  $k$ -dimensional nilpotent complex Lie group, which admits a natural fiberwise right action on  $J_k T_X$ . The action consists of reparametrizing  $k$ -jets of maps  $f : (\mathbb{C}, 0) \rightarrow X$  by a biholomorphic change of parameter  $\varphi : (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$  defined by  $(f, \varphi) \mapsto f \circ \varphi$ . The corresponding  $\mathbb{C}^*$ -action on  $k$ -jets is described in coordinates by

$$\lambda \cdot (f', f'', \dots, f^{(k)}) = (\lambda f', \lambda^2 f'', \dots, \lambda^k f^{(k)}).$$

Green-Griffiths introduced the vector bundle  $E_{k,m}^{\text{GG}} T_X^*$  whose fibres are complex valued polynomials  $Q(f', f'', \dots, f^{(k)})$  on the fibres of  $J_k T_X$ , of weighted degree  $m$  with respect to the  $\mathbb{C}^*$ -action, *i.e.*,  $Q(\lambda f', \lambda^2 f'', \dots, \lambda^k f^{(k)}) = \lambda^m Q(f', f'', \dots, f^{(k)})$ , for all  $\lambda \in \mathbb{C}^*$  and  $(f', f'', \dots, f^{(k)}) \in J_k V$ . One calls  $E_{k,m}^{\text{GG}} T_X^*$  the bundle of jet differentials of order  $k$  and weighted degree  $m$ . Let  $U \subset X$  be an open set with local coordinate  $(z_1, \dots, z_n)$ , then any local section  $P \in \Gamma(U, E_{k,m}^{\text{GG}} T_X^*|_U)$  can be written as

$$\sum_{|\alpha_1|+2|\alpha_2|+\dots+k|\alpha_k|=m} c_\alpha(z) (d^1 z)^{\alpha_1} (d^2 z)^{\alpha_2} \dots (d^k z)^{\alpha_k},$$

where  $c_\alpha(z) \in \Gamma(U, \mathcal{O}_U)$  for any  $\alpha := (\alpha_1, \dots, \alpha_k) \in (\mathbb{N}^n)^k$ , such that for any holomorphic curve  $\gamma : \Omega \rightarrow U$  with  $\Omega \subset \mathbb{C}$ , we have

$$P([\gamma])(t) = \sum_{|\alpha_1|+2|\alpha_2|+\dots+k|\alpha_k|=m} c_\alpha(\gamma(t)) (\gamma'(t))^{\alpha_1} (\gamma''(t))^{\alpha_2} \dots (\gamma^{(k)}(t))^{\alpha_k} \in \Gamma(\Omega, \mathcal{O}_\Omega),$$

where  $[\gamma] : \Omega \rightarrow J_k T_X|_U$  is the natural lifted holomorphic curve on  $J_k T_X$  induced by  $\gamma$ .

However, we are more interested in the more geometric context introduced by J.-P. Demailly in [Dem95]: the subbundle  $E_{k,m} V^* \subset E_{k,m}^{\text{GG}} V^*$  which is a set of polynomial differential operators  $Q(f, f'', \dots, f^{(k)})$  which are invariant under arbitrary changes of parametrization, that is, for any  $\varphi \in \mathbb{G}_k$ , we have

$$Q((f \circ \varphi)', (f \circ \varphi)'', \dots, (f \circ \varphi)^{(k)}) = \varphi'(0)^m Q(f', f'', \dots, f^{(k)}).$$

The bundle  $E_{k,m} V^*$  is called the bundle of *invariant jet differentials* of order  $k$  and degree  $m$ . A very natural construction for invariant jet differentials is *Wronskians*. In [Bro16] Brotbek introduced a type of Wronskians induced by global sections in some linear system. We will recall briefly his constructions in Section 2.3.

## 2.2 DEMAILLY-SEMPLÉ JET BUNDLES

Let  $X$  be a complex manifold of dimension  $n$ . If  $V$  is a subbundle of rank  $r$ , one constructs a tower of *Demailly-Semple  $k$ -jet bundles*  $\pi_{k-1,k} : (X_k, V_k) \rightarrow (X_{k-1}, V_{k-1})$  that are  $\mathbb{P}^{r-1}$ -bundles, with  $\dim X_k = n + k(r-1)$  and  $\text{rank}(V_k) = r$ . For this, we take  $(X_0, V_0) = (X, V)$ , and for every  $k \geq 1$ , inductively we set  $X_k := P(V_{k-1})$  and

$$V_k := (\pi_{k-1,k})_*^{-1} \mathcal{O}_{X_k}(-1) \subset T_{X_k},$$

where  $\mathcal{O}_{X_k}(1)$  is the tautological line bundle on  $X_k = P(V_{k-1})$ ,  $\pi_{k-1,k} : X_k \rightarrow X_{k-1}$  the natural projection and  $(\pi_{k-1,k})_* = d\pi_{k-1,k} : T_{X_k} \rightarrow \pi_{k-1,k}^* T_{X_{k-1}}$  its differential. By composing the projections we get for all pairs of indices  $0 \leq j \leq k$  natural morphisms

$$\pi_{j,k} : X_k \rightarrow X_j, \quad (\pi_{j,k})_* = (d\pi_{j,k})|_{V_k} : V_k \rightarrow (\pi_{j,k})^* V_j,$$

and for every  $k$ -tuple  $\mathbf{a} = (a_1, \dots, a_k) \in \mathbb{Z}^k$  we define

$$\mathcal{O}_{X_k}(\mathbf{a}) = \otimes_{1 \leq j \leq k} \pi_{j,k}^* \mathcal{O}_{X_j}(a_j).$$

We also have an inductively defined  $k$ -th lifting for germs of holomorphic curves such that  $f_{[k]} : (\mathbb{C}, 0) \rightarrow X_k$  is obtained as  $f_{[k]}(t) = (f_{[k-1]}(t), [f'_{[k-1]}(t)])$ . Moreover, if one denote by

$$J_k^{\text{reg}} V := \{[f]_k \in J_k V | f'(0) \neq 0\}$$



the space of *regular  $k$ -jets* tangent to  $V$ , then there exists a morphism

$$\begin{aligned} J_k^{\text{reg}} V &\rightarrow X_k \\ [f] &\mapsto f_{[k]}(0) \end{aligned}$$

whose image is an open set in  $X_k$  denote by  $X_k^{\text{reg}}$ , which can be identified with the quotient  $J_k^{\text{reg}}/\mathbb{G}_k$  [Dem95, Theorem 6.8]. In other words,  $X_k^{\text{reg}} \subset X_k$  is the set of elements  $f_{[k]}(0)$  in  $X_k$  which can be reached by all regular germs of curves  $f$ , and set  $X_k^{\text{sing}} := X_k \setminus X_k^{\text{reg}}$ , which is a divisor in  $X_k$ . Thus  $X_k$  is a relative compactification of  $J_k^{\text{reg}}/\mathbb{G}_k$  over  $X$ . Dealing with hyperbolicity problems, we are allowed to have small base locus contained in  $X_k^{\text{sing}}$  [Dem95, Section 7].

We will need the following parametrizing theorem due to J.-P. Demailly [Dem95, Corollary 5.12]:

**Theorem 2.1.** *Let  $(X, V)$  be a directed variety. For any  $w_0 \in X_k$ , there exists an open neighborhood  $U_{w_0}$  of  $w_0$  and a family of germs of curves  $(f_w)_{w \in U_{w_0}}$ , tangent to  $V$  depending holomorphically on  $w$  such that*

$$(f_w)_{[k]}(0) = w \quad \text{and} \quad (f_w)'_{[k-1]}(0) \neq 0, \quad \forall w \in U_{w_0}.$$

*In particular,  $(f_w)'_{[k-1]}(0)$  gives a local trivialization of the tautological line bundle  $\mathcal{O}_{X_k}(-1)$  on  $U_{w_0}$ .*

By [Dem95, Theorem 6.8], we have the following isomorphism between Demailly-Semple jet bundles and invariant jet differentials:

**Theorem 2.2.** *(Direct image formula) Let  $(X, V)$  be a directed variety. The direct image sheaf*

$$(\pi_{0,k})_* \mathcal{O}_{X_k}(m) \cong E_{k,m} V^* \tag{2.1}$$

*can be identified with the sheaf of holomorphic sections of  $E_{k,m} V^*$ . In particular, for any line bundle  $L$ , we have the following isomorphism induced by  $(\pi_{0,k})_*$ :*

$$(\pi_{0,k})_* : H^0(X_k, \mathcal{O}_{X_k}(m) \otimes \pi_{0,k}^* L) \xrightarrow{\cong} H^0(X, E_{k,m} V^* \otimes L). \tag{2.2}$$

*Moreover, let  $\mathbf{a} = (a_1, \dots, a_k) \in \mathbb{Z}^k$  and  $m = a_1 + \dots + a_k$ , then we have*

$$(\pi_{0,k})_* \mathcal{O}_{X_k}(\mathbf{a}) \cong \overline{F}^{\mathbf{a}} E_{k,m} V^* \tag{2.3}$$

*where  $\overline{F}^{\mathbf{a}} E_{k,m} V^*$  is the subbundle of polynomials  $Q(f', f'', \dots, f^{(k)}) \in E_{k,m} V^*$  involving only monomials  $(f^{(\bullet)})^l$  such that*

$$l_{s+1} + 2l_{s+2} + \dots + (k-s)l_k \leq a_{s+1} + \dots + a_k$$

*for all  $s = 0, \dots, k-1$ .*

Therefore, with the notations in Theorem 2.1, for any given local invariant jet differential  $P \in \Gamma(U, E_{k,m} V^*|_U)$ , the inverse image under  $(\pi_{0,k})_*$  is the section in

$$\sigma_P \in \Gamma(U_{w_0}, \mathcal{O}_{X_k}(m)|_{U_{w_0}})$$

defined by

$$\sigma_P(w) := P(f'_w, f''_w, \dots, f_w^{(k)}) ((f_w)'_{[k-1]}(0))^{-m}. \tag{2.4}$$

The general philosophy of the theory of (invariant) jet differentials is that their global sections with values in an anti-ample divisor provide algebraic differential equations which every entire curve must satisfy, which is an application of Ahlfors-Schwarz lemma. The following *Fundamental Vanishing Theorem* shows the obstructions to the existence of entire curves:

**Theorem 2.3.** *(Demailly, Green-Griffiths, Siu-Yeung) Let  $(X, V)$  be a directed projective variety and  $f : \mathbb{C} \rightarrow (X, V)$  an entire curve tangent to  $V$ . Then for every global section  $P \in H^0(X, E_{k,m} V^* \otimes \mathcal{O}(-A))$  where  $A$  is an ample divisor of  $X$ , one has  $P(f', f'', \dots, f^{(k)}) = 0$ . In other word, if we denote by  $s$  the unique section in  $H^0(X_k, \mathcal{O}_{X_k}(m) \otimes \pi_{0,k}^*(-A))$  corresponding to  $P$  induced by the isomorphism (2.2), and  $Z(s) \subset X_k$  the base locus of this section, then  $f_{[k]}(\mathbb{C}) \subset Z(s)$ .*

Now we state the following definition which describes the positivity of the invariant jet bundles:

**Definition 2.1.** *Let  $X$  be a projective manifold. We say that  $X$  has almost  $k$ -jet ampleness if and only if there exists a  $k$ -tuple of positive integers  $(a_1, \dots, a_k)$  such that  $\mathcal{O}_{X_k}(a_1, \dots, a_k)$  is big and its augmented base locus satisfies the condition*

$$\mathbb{B}_+(\mathcal{O}_{X_k}(a_1, \dots, a_k)) \subset X_k^{\text{sing}}.$$

By applying Theorem 2.3, we can quickly conclude that, if  $X$  has almost  $k$ -jet ampleness, then its Demailly-Semple locus [DR13, Section 2.1] is an empty set, and thus  $X$  is Kobayashi hyperbolic.

### 2.3 BROTBEEK'S WRONSKIANS

In this subsection, we will study the property of the Wronskians constructed by Brotbek in [Bro16], which associates any  $k+1$  sections of a given line bundle  $L$  to invariant  $k$ -jet differentials of weighted degree  $k' := \frac{(k+1)k}{2}$ , that is, sections in  $H^0(X, E_{k,k'}T_X^* \otimes L^{k+1})$ . We prove that, the Wronskians factorizes through a natural morphism from the bundle  $J^k \mathcal{O}_X(L)$  of  $k$ -jet sections of  $L$  to the invariant jet bundles  $E_{k,k'}T_X^* \otimes L^{k+1}$ . Moreover, we obtain an “effective finite generation” of the  $k$ -th Wronskian ideal sheaf  $\mathfrak{w}(X_k, L)$  (see also Theorem 2.4 below).

Let  $X$  be an  $n$ -dimensional compact complex manifold. If  $(z_1, \dots, z_n)$  are local holomorphic coordinates on an open set  $U \subset X$ , then since  $J_k T_X$  is a locally trivial holomorphic fiber bundle, we have the homeomorphism

$$J_k T_X|_U \sim U \times \mathbb{C}^{nk},$$

which is given by  $[f] \mapsto (f(0), f'(0), \dots, f^{(k)}(0))$ .

For any holomorphic function  $g \in \mathcal{O}(U)$ , and  $1 \leq j \leq k$ , there exists an induced holomorphic function  $d_U^{[j]}(g)$  on  $\mathcal{O}(p_k^{-1}(U))$ , defined by

$$d_U^{[j]}(g)(f'(0), f''(0), \dots, f^{(k)}(0)) := (g \circ f)^{(j)}(0).$$

Moreover, we have the following lemma

**Lemma 2.1.** *For any  $k \geq 1$ , we have  $d_U^{[k]}(g) \in \Gamma(U, E_{k,k}^{\text{GG}}T_U^*)$ , and*

$$d_U^{[k]}(g) = \sum_{|\alpha_1|+2|\alpha_2|+\dots+k|\alpha_k|=k} c_\alpha(z)(d^1 z)^{\alpha_1}(d^2 z)^{\alpha_2} \dots (d^k z)^{\alpha_k}, \quad (2.5)$$

such that for each  $\alpha := (\alpha_1, \dots, \alpha_k) \in (\mathbb{N}^n)^k$ ,  $c_\alpha(z) \in \Gamma(U, \mathcal{O}_U)$  is a  $\mathbb{Z}$ -linear combination of  $\frac{\partial^{|\beta|} g}{\partial z^\beta}(z)$  with  $|\beta| \leq k$ .

*Proof.* We will prove the lemma by induction on  $k$ . For  $k=1$ , we simply have

$$d_U^{[1]}(g) = \sum_{i=1}^n \frac{\partial g}{\partial z_i}(z) dz^i \in \Gamma(U, T_U^*),$$

and thus the statements are true for  $k=1$ .

Suppose that  $d_U^{[k]}(g)$  has the form (2.5), then we have

$$\begin{aligned} d_U^{[k+1]}(g) &= \sum_{|\alpha_1|+2|\alpha_2|+\dots+k|\alpha_k|=k} \sum_{i=1}^k \sum_{j=1}^n c_\alpha(z)(d^1 z)^{\alpha_1} \dots (d^i z)^{\alpha_i - \mathbf{e}_j} (d^{i+1} z)^{\alpha_{i+1} + \mathbf{e}_j} \dots (d^k z)^{\alpha_k} \\ &+ \sum_{|\alpha_1|+2|\alpha_2|+\dots+k|\alpha_k|=k} \sum_{j=1}^n \frac{\partial c_\alpha(z)}{\partial z^j} (d^1 z)^{\alpha_1 + \mathbf{e}_j} \dots (d^k z)^{\alpha_k}, \end{aligned}$$

where  $\mathbf{e}_1 = (1, 0, \dots, 0), \dots, \mathbf{e}_n = (0, \dots, 0, 1)$  is the standard basis in  $\mathbb{Z}^n$ . If the lemma is true for  $k$ , so is  $k+1$ . Thus the lemma holds for any  $k \in \mathbb{N}$ .  $\square$

Since the bundle

$$E_{k,\bullet}^{\text{GG}} T_X^* := \bigoplus_{m \geq 0} E_{k,m}^{\text{GG}} T_X^*$$

is a bundle of graded algebras (the product is obtained simply by taking the product of polynomials). There are natural inclusions  $E_{k,\bullet}^{\text{GG}} \subset E_{k+1,\bullet}^{\text{GG}}$  of algebras, hence

$$E_{\infty,\bullet}^{\text{GG}} T_X^* := \bigcup_{k \geq 0} E_{k,\bullet}^{\text{GG}} T_X^*$$

is also an (commutative) algebra. Then for any  $(k+1)$  holomorphic functions  $g_0, \dots, g_k \in \mathcal{O}(U)$ , one can associate them to a natural  $k$ -jet differentials of order  $k$  and weighted degree  $k' := \frac{k(k+1)}{2}$ , say *Wronskians*, in the following way

$$W_U(g_0, \dots, g_k) := \begin{vmatrix} d_U^{[0]}(g_0) & \cdots & d_U^{[0]}(g_k) \\ \vdots & \ddots & \vdots \\ d_U^{[k]}(g_0) & \cdots & d_U^{[k]}(g_k) \end{vmatrix} \in \Gamma(U, E_{k,k'}^{\text{GG}} T_U^*).$$

If we set

$$W_U(g_0, \dots, g_k) = \sum_{|\alpha_1|+2|\alpha_2|+\dots+k|\alpha_k|=k'} b_\alpha(z) (d^1 z)^{\alpha_1} (d^2 z)^{\alpha_2} \cdots (d^k z)^{\alpha_k},$$

then for each  $\alpha := (\alpha_1, \dots, \alpha_k) \in (\mathbb{N}^n)^k$  with  $|\alpha_1| + 2|\alpha_2| + \dots + k|\alpha_k| = k'$ , by Lemma 2.1 there exists  $\{a_{\alpha\beta} \in \mathbb{Z}\}_{\beta=(\beta_0, \dots, \beta_k), |\beta_j| \leq k}$ , such that we have

$$b_\alpha(z) = \sum_{|\beta_j| \leq k} a_{\alpha\beta} \frac{\partial^{|\beta_0|} g_0(z)}{\partial z^{\beta_0}} \cdots \frac{\partial^{|\beta_k|} g_k(z)}{\partial z^{\beta_k}}. \quad (2.6)$$

By the properties of the Wronskian, for any permutation  $\sigma \in \text{Sym}(\{0, 1, \dots, k\})$ , we always have

$$W_U(g_{\sigma(0)}, \dots, g_{\sigma(k)}) = (-1)^{\text{sign}(\sigma)} W_U(g_0, \dots, g_k),$$

and thus  $a_{\alpha\beta} = (-1)^{\text{sign}(\sigma)} a_{\alpha\sigma(\beta)}$ . Here  $\sigma(\beta) := (\beta_{\sigma(0)}, \dots, \beta_{\sigma(k)})$ .

On the other hand, for any holomorphic line bundle  $A$  on  $X$ , one can define the bundle  $J^k A$  of  $k$ -jet sections of  $A$  by  $(J^k A)_x = \mathcal{O}_x(A) / (\mathcal{M}_x^{k+1} \cdot \mathcal{O}_x(A))$  for every  $x \in X$ , where  $\mathcal{M}_x$  is the maximal ideal of  $\mathcal{O}_x$ . Then  $J^k A$  has a filtration whose graded bundle is  $\bigoplus_{0 \leq p \leq k} S^p T_X^* \otimes \mathcal{O}(A)$ . Set  $e_U$  to be a holomorphic frame of  $A$  and  $(z_1, \dots, z_n)$  analytic coordinates on an open subset  $U \subset X$ . The fiber  $(J^k A)_x$  can be identified with the set of Taylor developments of order  $k$ :

$$\sum_{|\alpha| \leq k} c_\alpha (z - x)^\alpha \cdot e_U,$$

and the coefficients  $c_\beta$  define coordinates along the fibers of  $J^k A$ . Thus one has a natural local trivialization of  $J^k A$  given by

$$\begin{aligned} \Psi_U : U \times \mathbb{C}^{I_k} &\rightarrow J^k A|_U, \\ (x, c_\beta) &\mapsto \sum_{\beta \in I_k} c_\beta (z - x)^\beta \cdot e_U, \end{aligned}$$

where

$$I_k := \{\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n \mid |\beta| \leq k\}.$$

For any local section  $s = s_U \cdot e_U \in \Gamma(U, A)$ , one has a natural map (no more a  $\mathcal{O}_U$ -module morphism!)

$$i_{[k]} : \Gamma(U, A) \rightarrow \Gamma(U, J^k A),$$



which is given by

$$\Psi_U^{-1} \circ i_{[k]}(s)(x) = (x, \frac{\partial^{|\alpha|} s_U}{\partial z^\alpha}(x)).$$

The local coordinates  $(z_1, \dots, z_n)$  on  $U$  also induces a natural local trivialization of the bundle of jet differentials  $E_{k,m}^{\text{GG}} T_U^* \rightarrow U$ . Indeed, as any local section of  $P \in \Gamma(U, E_{k,m}^{\text{GG}} T_X^*|_U)$  is given by

$$\sum_{|\alpha_1|+2|\alpha_2|+\dots+k|\alpha_k|=m} c_\alpha(z) (d^1 z)^{\alpha_1} (d^2 z)^{\alpha_2} \dots (d^k z)^{\alpha_k},$$

where  $c_\alpha(z) \in \Gamma(U, \mathcal{O}_U)$  for any  $\alpha$ , one has the natural local trivialization of  $E_{k,m}^{\text{GG}} T_X^* \rightarrow X$  given by

$$\begin{aligned} \Phi_U : U \times \mathbb{C}^{I_{k,m}} &\rightarrow E_{k,m}^{\text{GG}} T_U^*, \\ (z, c_\alpha) &\mapsto \sum_{|\alpha_1|+2|\alpha_2|+\dots+k|\alpha_k|=m} c_\alpha (d^1 z)^{\alpha_1} (d^2 z)^{\alpha_2} \dots (d^k z)^{\alpha_k}, \end{aligned}$$

where

$$I_{k,m} := \{\alpha = (\alpha_1, \dots, \alpha_k) \in (\mathbb{N}^n)^k \mid |\alpha_1| + 2|\alpha_2| + \dots + k|\alpha_k| = m\}.$$

Now we define a multi-linear map

$$\tilde{\mu} : \prod_{k=1}^{k+1} \mathbb{C}^{I_k} \rightarrow \mathbb{C}^{I_{k,k'}} \quad (2.7)$$

$$(c_{0,\beta_0}, \dots, c_{k,\beta_k}) \mapsto \left( \sum_{\beta=(\beta_0, \dots, \beta_k)} a_{\alpha\beta} c_{0,\beta_0} c_{1,\beta_1} \dots c_{k,\beta_k} \right)_{\alpha \in I_{k,k'}}, \quad (2.8)$$

where  $a_{\alpha\beta} \in \mathbb{Z}$  arises from (2.6). By the property that  $a_{\alpha\beta} = (-1)^{\text{sign}(\sigma)} a_{\alpha\sigma(\beta)}$  for any permutation  $\sigma$ , the multi-linear map  $\tilde{\mu}$  is alternating, and thus there exists a unique linear map

$$\mu : \wedge^{k+1} \mathbb{C}^{I_k} \rightarrow \mathbb{C}^{I_{k,k'}},$$

such that  $\tilde{\mu} = \mu \circ w$ . Here the map

$$w : \prod_{k=1}^{k+1} \mathbb{C}^{I_k} \rightarrow \wedge^{k+1} \mathbb{C}^{I_k}$$

which associates to  $k+1$  vectors from  $\mathbb{C}^{I_k}$  their exterior product.

By the local trivialization  $\Psi_U$  and  $\Phi_U$ ,  $\mu$  induces a bundle morphism

$$\tilde{W}_U : \wedge^{k+1} (J^k \mathcal{O}_U) \rightarrow E_{k,k'} T_U^*$$

defined by

$$\begin{array}{ccc} U \times \wedge^{k+1} \mathbb{C}^{I_k} & \xrightarrow{\mathbb{1} \times \mu} & U \times \mathbb{C}^{I_{k,k'}} \\ \downarrow \Psi_U & & \downarrow \Phi_U \\ \wedge^{k+1} J^k \mathcal{O}_U & \xrightarrow{\tilde{W}_U} & E_{k,k'}^{\text{GG}} T_U^*. \end{array}$$

Composing with  $i_{[k]} : \Gamma(U, \mathcal{O}_U) \rightarrow \Gamma(U, J^k \mathcal{O}_U)$ , we recover Brotbek's Wronskians  $W_U$

$$W_U : \wedge^{k+1} H^0(U, \mathcal{O}_U) \xrightarrow{i_{[k]}} \wedge^{k+1} H^0(U, J^k \mathcal{O}_U) \rightarrow H^0(U, \wedge^{k+1} J^k \mathcal{O}_U) \xrightarrow{\tilde{W}_U} H^0(U, E_{k,k'}^{\text{GG}} T_U^*).$$

An important fact for the Wronskian is that, it is invariant under the  $\mathbb{G}_k$  action [Bro16, Proposition 2.2]:

**Lemma 2.2.** *With the notation as above,  $W_U(g_0, \dots, g_k) \in E_{k,k'} T_U^*$ , where  $k' := \frac{k(k+1)}{2}$ .*

In other words, the bundle morphism  $\tilde{W}_U$  factors through the subbundle

$$E_{k,k'}T_U^* \subset E_{k,k'}^{\text{GG}}T_U^*.$$

Now we consider the Demailly-Semple  $k$ -jet bundle of  $(X_k, V_k)$  of the direct variety  $(X, T_X)$  constructed in Section 2.2. Fix coordinates  $(z_1, \dots, z_n)$  on  $U$ ,  $T_X|_U$  can be trivialized with the basis  $\{\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}\}$ . Set  $U_k := X_k \cap \pi_{0,k}^{-1}(U)$ , and under that trivialization we have

$$U_k = U \times \mathcal{R}_{n,k},$$

where  $\mathcal{R}_{n,k}$  is some rational variety introduced in [Dem95, Theorem 9.1]. Moreover, the tautological bundle

$$\mathcal{O}_{X_k}(1)|_{U_k} = \text{pr}_2^*(\mathcal{O}_{\mathcal{R}_{n,k}}(1)), \quad (2.9)$$

where  $\text{pr}_2 : U_k \rightarrow \mathcal{R}_{n,k}$  is the projection on the factor  $\mathcal{R}_{n,k}$ . By the direct image formula (2.1)

$$(\pi_{0,k})_* \mathcal{O}_{X_k}(m) \cong E_{k,m}T_X^*,$$

we conclude that, under the above trivialization, the direct image  $(\pi_{0,k})_*$  induces a natural isomorphism (or a local trivialization of the vector bundle  $E_{k,m}T_U^*$ )

$$\varphi_U : U \times H^0(\mathcal{R}_{n,k}, \mathcal{O}_{\mathcal{R}_{n,k}}(m)) \rightarrow E_{k,m}T_U^*. \quad (2.10)$$

Moreover, under the trivialization  $\Phi_U$ , the inclusion  $E_{k,m}T_X^* \subset E_{k,m}^{\text{GG}}T_X^*$  is also a constant linear injective map, that is, there exists an injective linear map  $\nu : F^{k,m} \rightarrow \mathbb{C}^{I_{k,m}}$  such that

$$\begin{array}{ccc} U \times F^{k,m} & \xrightarrow{1 \times \nu} & U \times \mathbb{C}^{I_{k,m}} \\ \downarrow \varphi_U & & \downarrow \Phi_U \\ E_{k,m}T_X^*|_U & \hookrightarrow & E_{k,m}^{\text{GG}}T_X^*|_U. \end{array}$$

Here we denote  $F^{k,m} := H^0(\mathcal{R}_{n,k}, \mathcal{O}_{\mathcal{R}_{n,k}}(m))$ .

Therefore, under the trivializations  $\varphi_U$  and  $\Psi_U$ , the factorised bundle morphism  $\tilde{W}_U$  is still a constant linear map. That is, there exists a linear map  $\tilde{\nu} : \wedge^{k+1} \mathbb{C}^{I_k} \rightarrow F^{k,k'}$  such that  $\mu = \nu \circ \tilde{\nu}$  and we have the following diagram:

$$\begin{array}{ccccc} U \times \wedge^{k+1} \mathbb{C}^{I_k} & \xrightarrow{1 \times \tilde{\nu}} & U \times F^{k,k'} & \xrightarrow{1 \times \nu} & U \times \mathbb{C}^{I_{k,m}} \\ \downarrow \Psi_U & & \downarrow \varphi_U & & \downarrow \Phi_U \\ \wedge^{k+1} J^k \mathcal{O}_U & \xrightarrow{\tilde{W}_U} & E_{k,k'}T_U^* & \hookrightarrow & E_{k,m}^{\text{GG}}T_X^*|_U. \end{array}$$

We set

$$S := \text{Image}(\tilde{\nu}) \subset H^0(\mathcal{R}_{n,k}, \mathcal{O}_{\mathcal{R}_{n,k}}(k')),$$

and denote by  $\mathcal{I}_{n,k} \subset \mathcal{O}_{\mathcal{R}_{n,k}}$  the base ideal of the linear system  $S$ . Denote  $\mathfrak{w}_U$  to be the ideal sheaf  $\text{pr}_2^*(\mathcal{I}_{n,k})$  on  $U_k$ .

On the other hand, one has a natural global construction for the invariant jet differentials on  $X$ : let  $L$  be any holomorphic line bundle on  $X$ , for any  $s_0, \dots, s_k \in H^0(X, L)$ , if we choose a local trivialization of  $L$  above  $U$ , we define

$$W_U(s_0, \dots, s_k) := W_U(s_{0,U}, \dots, s_{k,U}) \in \Gamma(U, E_{k,k'}T_U^*),$$

and if gluing together, we have the global section [Bro16, Proposition 2.3]:

**Proposition 2.1.** *For any  $s_0, \dots, s_k \in H^0(X, L)$ , the locally defined jet differential equations  $W_U(s_0, \dots, s_k)$  glue together into a global section*

$$W(s_0, \dots, s_k) \in H^0(X, E_{k,k'} T_X^* \otimes L^{k+1}).$$

The proof of the proposition follows from the fact that for any  $s_U \in \Gamma(U, \mathcal{O}_U)$ , we have

$$W_U(s_U s_{0,U}, \dots, s_U s_{k,U}) = s_U^{k+1} W_U(s_{0,U}, \dots, s_{k,U}).$$

We will denote by

$$\omega(s_0, \dots, s_k) = (\pi_{0,k})_*^{-1} W(s_0, \dots, s_k) \in H^0(X_k, \mathcal{O}_{X_k}(k') \otimes \pi_{0,k}^* L^{k+1}) \quad (2.11)$$

the inverse image of the Wronskian  $W(s_0, \dots, s_k)$  under the global isomorphism (2.2) induced by the direct image  $(\pi_{0,k})_*$ .

Now let

$$\mathbb{W}(X_k, L) := \text{Span}\{\omega(s_0, \dots, s_n) | s_0, \dots, s_n \in H^0(X, L)\} \subset H^0(X_k, \mathcal{O}_{X_k}(k') \otimes \pi_{0,k}^* L^{k+1})$$

be the associated sublinear system of  $H^0(X_k, \mathcal{O}_{X_k}(k') \otimes \pi_{0,k}^* L^{k+1})$ . One defines the  $k$ -th Wronskian ideal sheaf of  $L$ , denoted by  $\mathfrak{w}(X_k, L)$ , to be the base ideal  $\mathfrak{b}(\mathbb{W}(X_k, L))$  of the linear system  $\mathbb{W}(X_k, L)$ .

By the definition, if  $A$  is any line bundle on  $X$ , and  $s \in H^0(X, A)$ , we have

$$W(s \cdot s_0, \dots, s \cdot s_k) = s^{k+1} W(s_0, \dots, s_k) \in H^0(X, E_{k,k'} T_X^* \otimes L^{k+1} \otimes A^{k+1}).$$

Thus if  $L$  is very ample we have a chain of inclusions

$$\mathfrak{w}(X_k, L) \subset \mathfrak{w}(X_k, L^2) \subset \dots \subset \mathfrak{w}(X_k, L^m) \subset \dots$$

By the Noetherianity, this increasing sequence stabilizes after some  $m_\infty(X_k, L)$ , and we denote the obtained asymptotic ideal by

$$\mathfrak{w}_\infty(X_k, L) := \mathfrak{w}(X_k, L^m) \quad \text{for any } m \geq m_\infty(X_k, L). \quad (2.12)$$

An important property for  $\mathfrak{w}(X_k, L)$  is the following in [Bro16, Lemma 2.4]:

**Lemma 2.3.** *If  $L$  generates  $k$ -jets at every point of  $X$ , that is, for any  $x \in X$ , the map*

$$H^0(X, L) \rightarrow L \otimes \mathcal{O}_{X,x} / \mathcal{M}_{X,x}^{k+1} = (J^k L)_x$$

*is surjective, where  $\mathcal{M}_x$  is the maximal ideal of  $\mathcal{O}_x$ , then*

$$\text{Supp}(\mathcal{O}_{X_k} / \mathfrak{w}(X_k, L)) \subset X_k^{\text{sing}}.$$

For any very ample line bundle  $L$ , assume that  $L|_U$  can be trivialized. Now we will compare the globally defined asymptotic Wronskian ideal sheaf  $\mathfrak{w}_\infty(X_k, L)$  with our locally defined  $\mathfrak{w}_U$ .

When restricted to  $U_k := \pi_{0,k}^{-1}(U)$ , the global map

$$\omega(\bullet) : \wedge^{k+1} H^0(X, \mathcal{O}_X(L)) \rightarrow H^0(X_k, \mathcal{O}_{X_k}(k') \otimes \pi_{0,k}^* L^{k+1})$$

defined in (2.11) can be localized as the following

$$\begin{array}{ccccccc} \omega_U : \wedge^{k+1} H^0(X, \mathcal{O}_X(L)) & \xrightarrow{i_{[k]}} & \wedge^{k+1} H^0(U, J^k L|_U) & \longrightarrow & H^0(U, \wedge^{k+1} J^k \mathcal{O}_U) & \xrightarrow{\tilde{W}_U} & H^0(U, E_{k,k'} T_U^*) \\ \downarrow \mathbb{1} & & \Downarrow \Psi_U^{-1} & & \Downarrow \Psi_U^{-1} & & \Downarrow \varphi_U^{-1} \\ \wedge^{k+1} H^0(X, \mathcal{O}_X(L)) & \xrightarrow{\Psi_U^{-1} \circ i_{[k]}} & \wedge^{k+1} H^0(U, U \times \mathbb{C}^{I_k}) & \xrightarrow{l_k} & H^0(U, U \times \wedge^{k+1} \mathbb{C}^{I_k}) & \xrightarrow{\tilde{v}} & H^0(U, U \times F^{k,k'}). \end{array}$$

where  $H^0(U, U \times \wedge^{k+1} \mathbb{C}^{I_k})$ ,  $H^0(U, U \times \mathbb{C}^{I_k})$  and  $H^0(U, U \times F^{k,k'})$  are the sets of sections of the trivial bundles, and we also use the relation  $\mathcal{O}_{X_k}(k')|_{U_k} = \text{pr}_2^*(\mathcal{O}_{\mathcal{R}_{n,k}}(k'))$  in (2.9) to identify

$$H^0(U, U \times F^{k,k'}) \cong H^0(U_k, \mathcal{O}_{X_k}(k')|_{U_k}).$$

Then by the definition we have

$$\mathfrak{w}(X_k, L)|_{U_k} = \mathfrak{b}(\{\omega_U(s_0 \wedge \dots \wedge s_k)|_{s_0, \dots, s_k \in H^0(X, L)}\}).$$

Now we choose arbitrary sections  $s_0, \dots, s_k \in H^0(X, \mathcal{O}_X(L))$ , we have

$$h(s_0 \wedge \dots \wedge s_k) := l_k \circ \Psi_U^{-1} \circ i_{[k]}(s_0 \wedge \dots \wedge s_k) \in \Gamma(U, U \times \wedge^{k+1} \mathbb{C}^{I_k}),$$

which is a holomorphic section of the trivial bundle  $U \times \wedge^{k+1} \mathbb{C}^{I_k} \rightarrow U$ . Thus

$$\omega_U(s_0 \wedge \dots \wedge s_k) = \tilde{\nu} \circ h(s_0 \wedge \dots \wedge s_k)$$

is a holomorphic section of the trivial bundle  $U \times F^{k,k'} \rightarrow U$ , where  $\tilde{\nu} : \wedge^{k+1} \mathbb{C}^{I_k} \rightarrow F^{k,k'}$  is a  $\mathbb{C}$ -linear map.

Recall that

$$\mathcal{I}_{n,k} := \mathfrak{b}(\text{Image}(\tilde{\nu})) \subset \mathcal{O}_{\mathcal{R}_{n,k}}$$

is the base ideal of the linear system  $\text{Image}(\tilde{\nu}) \subset H^0(\mathcal{R}_{n,k}, \mathcal{O}_{\mathcal{R}_{n,k}}(k'))$ , and  $\mathfrak{w}_U$  is defined to be the ideal sheaf  $\text{pr}_2^*(\mathcal{I}_{n,k})$  on  $U_k$ . Thus the zero scheme of  $\tilde{\nu} \circ h(s_0 \wedge \dots \wedge s_k)$  is contained in  $\mathfrak{w}_U$ . As  $s_0, \dots, s_k$  are arbitrary, we always have

$$\mathfrak{w}(X_k, L)|_{U_k} \subset \mathfrak{w}_U. \quad (2.13)$$

On the other hand, suppose that the line bundle  $L$  generates  $k$ -jets, *i.e.*, the  $\mathbb{C}$ -linear map

$$H^0(X, L) \rightarrow (J^k L)_x$$

is surjective for any  $x \in X$ . Then for any  $x \in U$ , any vector  $e \in \wedge^{k+1} \mathbb{C}^{I_k}$ , there exists  $r(k+1)$  sections  $\{s_{ji}\}_{0 \leq j \leq k, 1 \leq i \leq r} \in H^0(X, L)$  such that

$$e = \sum_{i=1}^r h(s_{0i} \wedge \dots \wedge s_{ki})(x).$$

Therefore, the set of images  $\omega_U(\bullet)(x) = F^{k,k'}$  for any  $x \in U$ , and thus the ideal sheaf  $\mathfrak{w}(X_k, L)$ , when restricted to each fiber  $x \times \mathcal{R}_{n,k} \subset U_k$ , is equal to  $\mathcal{I}_{n,k}$ . That is, if we denote by

$$i_x : \mathcal{R}_{n,k} \rightarrow U_k$$

which is induced the inclusive map  $x \times \mathcal{R}_{n,k} \rightarrow U \times \mathcal{R}_{n,k}$ , the inverse image of  $\mathfrak{w}(X_k, L)$  under  $i_x$

$$i_x^*(\mathfrak{w}(X_k, L)) := i_x^{-1} \mathfrak{w}(X_k, L) \otimes_{i_x^{-1} \mathcal{O}_{U_k}} \mathcal{O}_{\mathcal{R}_{n,k}}$$

is the same as  $\mathcal{I}_{n,k}$ . Thus we have

$$\mathfrak{w}(X_k, L)|_{U_k} = \mathfrak{w}_U. \quad (2.14)$$

As  $U$  is any open set on  $X$  with local coordinates  $(z_1, \dots, z_n)$  such that  $L|_U$  can be trivialized, from the inclusive relation (2.13) we see that

$$\mathfrak{w}(X_k, L) = \mathfrak{w}(X_k, L^2) = \dots = \mathfrak{w}(X_k, L^k) = \dots,$$

and thus we conclude that, for any ample line bundle  $L$  which generates  $k$ -jets everywhere, the  $k$ -th Wronskian ideal sheaf of  $L$  coincides with the asymptotic ideal sheaf

$$\mathfrak{w}(X_k, L) = \mathfrak{w}_\infty(X_k, L).$$

Moreover, from the local relation (2.14), we see that this asymptotic ideal sheaf does not depend on the choice of the very ample line bundle  $L$ , which was also proved by Brotbek in [Bro16, Lemma 2.6]. We denote by  $\mathfrak{w}_\infty(X_k)$  the asymptotic Wronskian ideal sheaf.

In conclusion, we have the following theorem:

**Theorem 2.4.** *If  $L$  generates  $k$ -jets at each point of  $X$ , then  $\mathfrak{w}(X_k, L) = \mathfrak{w}_\infty(X_k)$  and  $m_\infty(X_k, L) = 1$ . In particular, if  $L$  is known to be only very ample, we have  $\mathfrak{w}(X_k, L^k) = \mathfrak{w}_\infty(X_k)$  and  $m_\infty(X_k, L) = k$ .*

As was shown in [Bro16, Lemma 2.6],  $\mathfrak{w}_\infty(X_k)$  behaves well under restriction, that is, for any directed variety  $(Y, V_Y)$  with  $Y \subset X$  and  $V_Y \subset V_X|_Y$ , under the induced inclusion  $Y_k \subset X_k$  one has

$$\mathfrak{w}_\infty(X_k)|_{Y_k} = \mathfrak{w}_\infty(Y_k).$$

## 2.4 BLOW-UPS OF THE WRONSKIAN IDEAL SHEAF

This subsection are mainly borrowed from [Bro16]. We will state some important results without proof, and the readers who are interested in the details are encouraged to refer to [Bro16, Section 2.4].

From [Dem95, Theorem 6.8],  $\mathcal{O}_{X_k}(1)$  is only relatively big, and  $X_k^{\text{Sing}}$  is the obstruction to the ampleness of  $\mathcal{O}_{X_k}(1)$ . However, for the hyperbolicity problems,  $X_k^{\text{Sing}}$  is negligible since  $X_k$  is a relative compactification of  $J_k^{\text{reg}}/\mathbb{G}_k = X_k^{\text{reg}}$  over  $X$ , and for every non-constant entire curve  $f$  on  $X$ , its  $k$ -th lift  $f_{[k]} : \mathbb{C} \rightarrow X_k$  can not be contained in  $X_k^{\text{Sing}}$ . Thus we want to find a *good and functorial* compactification of  $X_k^{\text{reg}}$  such that the tautological line bundle is ample. Brotbek introduced a clever way to overcome this difficulty.

For any directed manifold  $(X, V)$ , we denote by

$$\hat{X}_k := \text{Bl}_{\mathfrak{w}_\infty(X_k)}(X_k) \rightarrow X_k$$

the blow-up of  $X_k$  along  $\mathfrak{w}_\infty(X_k)$ , and  $F$  the effective Cartier divisor on  $\hat{X}_k$  such that

$$\mathcal{O}_{\hat{X}_k}(-F) = \nu_k^{-1} \mathfrak{w}_\infty(X_k).$$

Take a very ample line bundle  $L$  on  $X$ , for any  $m \geq 0$ , and any  $s_0, \dots, s_k \in H^0(X, L^m)$ , there exists

$$\hat{\omega}(s_0, \dots, s_k) \in H^0\left(\hat{X}_k, \nu_k^*(\mathcal{O}_{X_k}(k') \otimes \pi_{0,k}^* L^{m(k+1)}) \otimes \mathcal{O}_{\hat{X}_k}(-F)\right),$$

such that

$$\nu_k^* \omega(s_0, \dots, s_k) = s_F \cdot \hat{\omega}(s_0, \dots, s_k).$$

Here  $s_F \in H^0(\hat{X}_k, F)$  is the tautological section. Then by Theorem 2.4, for any  $\hat{w} \in \hat{X}_k$  and any  $m \geq k$ , there exists  $s_0, \dots, s_k \in H^0(X, L^m)$  such that

$$\hat{\omega}(s_0, \dots, s_k)(\hat{w}) \neq 0.$$

The blow-ups is functorial thanks to the fact that the asymptotic Wronskian ideal sheaf behaves well under restriction. Namely, if  $(Y, V_Y) \subset (X, V_X)$  is a sub-directed variety, then  $\hat{Y}_k$  is the strict transform of  $Y_k$  in  $X_k$  under the blowing-up morphism  $\nu_k : \hat{X}_k \rightarrow X_k$ . This functorial property also holds for families [Bro16, Proposition 2.7]:

**Theorem 2.5.** *Let  $\mathcal{X} \xrightarrow{\rho} T$  be a smooth and projective morphism between non-singular varieties. We denote by  $\mathcal{X}_k^{\text{rel}}$  the  $k$ -th Demailly-Semple tower of the relative directed variety  $(\mathcal{X}, T_{\mathcal{X}/T})$ . Take  $\nu_k : \hat{\mathcal{X}}_k^{\text{rel}} \rightarrow \mathcal{X}_k^{\text{rel}}$  to be the blow-ups of the asymptotic Wronskian ideal sheaf  $\mathfrak{w}_\infty(\mathcal{X}_k^{\text{rel}})$ . Then for any  $t_0 \in T$  writing  $X_{t_0} := \rho^{-1}(t_0)$ , we have*

$$\nu_k^{-1}(X_{t_0,k}) = \hat{X}_{t_0,k}.$$

## 3 PROOF OF THE MAIN THEOREMS

### 3.1 FAMILIES OF COMPLETE INTERSECTIONS OF FERMAT-TYPE HYPERSURFACES

Let  $X$  be a projective manifold of dimension  $n$  endowed with a very ample line bundle  $A$ . We first construct a family of complete intersection subvarieties in  $X$  cut out by certain Fermat-type

hypersurfaces. For an integer  $N \geq n$ , we fix  $N + 1$  sections in general position  $\tau_0, \dots, \tau_N \in H^0(X, A)$ . By “general position” we mean that the hypersurfaces  $\{\tau_i = 0\}_{i=0, \dots, N}$  are all smooth and irreducible ones, and they are simple normal crossing. For any  $1 \leq c \leq n - 1$ , and two  $c$ -tuples of positive integers  $\epsilon = (\epsilon_1, \dots, \epsilon_c), \delta = (\delta_1, \dots, \delta_c)$ , we construct the family  $\mathcal{X}$  as follows: For any  $p = 1, \dots, c$ , set  $\mathbb{I}^p := \{I = (i_0, \dots, i_N) \mid |I| = \delta_p\}$  and  $\mathbf{a}^p := \left( a_I^p \in H^0(X, \mathcal{O}_X(\epsilon_p A)) \right)_{|I|=\delta_p}$ . For the positive integers  $r$  and  $k$  fixed later according to our needs, we define the bihomogenous sections of  $\mathcal{O}_X((\epsilon_p + (r+k)\delta_p)A)$  over  $X$  by

$$\mathbf{F}^p(\mathbf{a}^p)(x) : x \mapsto \sum_{|I|=\delta_p} a_I^p(x) \tau(x)^{(r+k)I},$$

where  $\mathbf{a}^p$  varies in the parameter space  $S_p := \bigoplus_{I \in \mathbb{I}^p} H^0(X, \mathcal{O}_X(\epsilon_p A))$ , and  $\tau := (\tau_0, \dots, \tau_N)$ .

We then consider the family  $\overline{\mathcal{X}} \subset S_1 \times \dots \times S_c \times X$  of complete intersection varieties in  $X$  defined by those sections:

$$\overline{\mathcal{X}} := \{(\mathbf{a}^1, \dots, \mathbf{a}^c, x) \in S_1 \times \dots \times S_c \times X \mid \mathbf{F}^1(\mathbf{a}^1)(x) = \dots = \mathbf{F}^c(\mathbf{a}^c)(x) = 0\}. \quad (3.1)$$

We know that there is a non-empty Zariski open set  $S \subset S_1 \times \dots \times S_c$  parametrizing smooth varieties and we will work on  $\mathcal{X} := q_1^{-1}(S) \cap \overline{\mathcal{X}}$ , where  $q_1$  is the natural projection from  $S_1 \times \dots \times S_c \times X$  to  $S_1 \times \dots \times S_c$ . Set  $\mathcal{X}_k$  to be the  $k$ -th Demailly-Semple tower of the relative tangent bundle  $(\mathcal{X}, T_{\mathcal{X}/S})$ , and  $\hat{\mathcal{X}}_k$  the blowing-up of the asymptotic Wronskian ideal sheaf  $\mathfrak{w}_\infty(\mathcal{X}_k)$ , and we would like to construct a regular morphism from  $\hat{\mathcal{X}}_k$  (after shrinking a bit) to a suitable generically finite to one family and to “pull-back” the positivity from the parameter space of this family. First we begin with a technical lemma by Brotbek [Bro16, Lemma 3.2]:

**Lemma 3.1.** *Let  $U$  be an open subset of  $X$  on which both  $A$  and  $T_X$  can be trivialized. Fix any  $1 \leq p \leq c$ . For any  $I = (i_0, \dots, i_N) \in \mathbb{I}^p$ , there exists a  $\mathbb{C}$ -linear map*

$$d_{I,U}^{[j]} : H^0(X, \epsilon_p A) \rightarrow \Gamma(U, E_{k,k}^{\text{GG}} T_U^*)$$

such that for any  $a \in H^0(X, \epsilon_p A)$ ,  $d_U^{[j]}(a \tau^{(r+k)I}) = \tau_U^{rI} d_{I,U}^{[j]}(a)$ , where  $\tau_U := (\tau_{0,U}, \dots, \tau_{N,U})$  is the local trivialization of  $\tau$  over  $U$ .

Therefore, for any  $I_0, \dots, I_k \in \mathbb{I}^p$  and any  $a_{I_0}, \dots, a_{I_k} \in H^0(X, \epsilon_p A)$  one can define

$$W_{U, I_0, \dots, I_k}(a_{I_0}, \dots, a_{I_k}) := \begin{vmatrix} d_{I_0,U}^{[0]}(a_{I_0}) & \dots & d_{I_k,U}^{[0]}(a_{I_k}) \\ \vdots & \ddots & \vdots \\ d_{I_0,U}^{[k]}(a_{I_0}) & \dots & d_{I_k,U}^{[k]}(a_{I_k}) \end{vmatrix} \in \Gamma(U, E_{k,k'}^{\text{GG}} T_U^*), \quad (3.2)$$

and by Lemma 3.1 we obtain

$$W_U(a_{I_0} \tau^{(r+k)I_0}, \dots, a_{I_k} \tau^{(r+k)I_k}) = \tau_U^{r(I_0 + \dots + I_k)} W_{U, I_0, \dots, I_k}(a_{I_0}, \dots, a_{I_k}).$$

From Proposition 2.1 one can also glue them together

**Lemma 3.2.** *For any  $I_0, \dots, I_k \in \mathbb{I}^p$  and any  $a_{I_0}, \dots, a_{I_k} \in H^0(X, \epsilon_p A)$ , the locally defined functions  $W_{U, I_0, \dots, I_k}(a_{I_0}, \dots, a_{I_k})$  can be glued together into a global section*

$$W_{I_0, \dots, I_k}(a_{I_0}, \dots, a_{I_k}) \in H^0(X, E_{k,k'}^{\text{GG}} T_X^* \otimes A^{(k+1)(\epsilon_p + k\delta_p)})$$

such that

$$W(a_{I_0} \tau^{(r+k)I_0}, \dots, a_{I_k} \tau^{(r+k)I_k}) = \tau^{r(I_0 + \dots + I_k)} W_{I_0, \dots, I_k}(a_{I_0}, \dots, a_{I_k}). \quad (3.3)$$



We denote by

$$\omega_{I_0, \dots, I_k}(a_{I_0}, \dots, a_{I_k}) \in H^0(X_k, \mathcal{O}_{X_k}(k') \otimes \pi_{0,k}^* A^{(k+1)(\epsilon_p + k\delta_p)}) \quad (3.4)$$

the inverse image of  $W_{I_0, \dots, I_k}(a_{I_0}, \dots, a_{I_k})$  under the isomorphism (2.2), then by (3.3) we have

$$\omega(a_{I_0} \tau^{(r+k)I_0}, \dots, a_{I_k} \tau^{(r+k)I_k}) = (\pi_{0,k}^* \tau)^{r(I_0 + \dots + I_k)} \omega_{I_0, \dots, I_k}(a_{I_0}, \dots, a_{I_k}). \quad (3.5)$$

Hence for every  $1 \leq p \leq c$  we can construct a rational map given by the Wronskians

$$\begin{aligned} \Phi^p : S_p \times X_k &\dashrightarrow P(\Lambda^{k+1} \mathbb{C}^{\mathbb{I}^p}) \\ (\mathbf{a}, w) &\mapsto ([\omega_{I_0, \dots, I_k}(a_{I_0}, \dots, a_{I_k})(w)])_{I_0, \dots, I_k \in \mathbb{I}^p}, \end{aligned}$$

where  $\mathbb{C}^{\mathbb{I}^p} := \bigoplus_{I \in \mathbb{I}^p} \mathbb{C} \simeq \mathbb{C}^{\binom{N+\delta_p}{\delta_p}}$ .

**Claim 3.1.**  $\Phi^p$  factors through the Plücker embedding

$$\text{Pluc} : \text{Gr}_{k+1}(\mathbb{C}^{\mathbb{I}^p}) \hookrightarrow P(\Lambda^{k+1} \mathbb{C}^{\mathbb{I}^p}).$$

Proof: For any  $w_0 \in X_k$ , by Theorem 2.1, one can find an open neighborhood  $U_{w_0}$  of  $w_0$  with  $U_{w_0} \subset \pi_{0,k}^{-1}(U)$ , where  $A|_U$  can be trivialized; and a family of germs of curves  $(f_w)_{w \in U_{w_0}}$  depending holomorphically on  $w$  with  $(f_w)_{[k]}(0) = w$ . Then for any  $\mathbf{a} = (a_I)_{I \in \mathbb{I}^p} \in S_p$  and any  $0 \leq j \leq k$ , we denote by

$$d_{\bullet, w_0}^{[j]}(\mathbf{a}, w) := (d_{I, U}^{[j]}(a_I)(f'_w, f''_w, \dots, f_w^{(k)}))_{I \in \mathbb{I}^p} \in \mathbb{C}^{\mathbb{I}^p},$$

and the local rational map

$$\begin{aligned} \Phi_{w_0}^p : S_p \times U_{w_0} &\dashrightarrow \text{Gr}_{k+1}(\mathbb{C}^{\mathbb{I}^p}) \\ (\mathbf{a}, w) &\mapsto \text{Span}(d_{\bullet, w_0}^{[0]}(\mathbf{a}, w), \dots, d_{\bullet, w_0}^{[k]}(\mathbf{a}, w)). \end{aligned} \quad (3.6)$$

We will show that this definition does not depend on the choice of  $w_0$ . Indeed, by Definition 3.2 one has  $\Phi^p = \text{Pluc} \circ \Phi_{w_0}^p$ , which shows that  $\Phi^p$  factor through Pluc and we still denote by  $\Phi^p : S_p \times X_k \dashrightarrow \text{Gr}_{k+1}(\mathbb{C}^{\mathbb{I}^p})$  by abuse of notation.  $\blacksquare$

Recall that  $\hat{X}_k$  is denoted to be the blow-up  $\nu_k : \hat{X}_k \rightarrow X_k$  of the asymptotic  $k$ -th Wronskian ideal sheaf  $\mathfrak{w}_\infty(X_k)$ , such that  $\nu_k^{-1} \mathfrak{w}_\infty(X_k) = \mathcal{O}_{\hat{X}_k}(-F)$  for some effective cartier divisor  $F$  on  $\hat{X}_k$ . First, we have the following

**Claim 3.2.**  $\hat{\nu}_k$  induces a rational map

$$\hat{\Phi}^p : S_p \times \hat{X}_k \dashrightarrow \text{Gr}_{k+1}(\mathbb{C}^{\mathbb{I}^p}),$$

such that

$$\begin{array}{ccc} S_p \times \hat{X}_k & & \\ \downarrow \scriptstyle 1 \times \nu_k & \searrow \scriptstyle \hat{\Phi}^p & \\ S_p \times X_k & \xrightarrow{\scriptstyle \Phi^p} & \text{Gr}_{k+1}(\mathbb{C}^{\mathbb{I}^p}) \end{array}$$

Proof: By the definition of the asymptotic Wronskian ideal sheaf  $\mathfrak{w}_\infty(X_k)$ , we have

$$\omega(a_{I_0} \tau^{(r+k)I_0}, \dots, a_{I_k} \tau^{(r+k)I_k}) \in H^0(X_k, \mathcal{O}_{X_k}(k') \otimes \pi_{0,k}^* A^{(k+1)(\epsilon_p + (k+r)\delta_p)} \otimes \mathfrak{w}_\infty(X_k)).$$

Since  $(\pi_{0,k}^* \tau)^{r(I_0 + \dots + I_k)}$  does not vanish along any irreducible component of the zero scheme of  $\mathfrak{w}_\infty(X_k)$ , by (3.5) we see that

$$\omega_{I_0, \dots, I_k}(a_{I_0}, \dots, a_{I_k}) \in H^0(X_k, \mathcal{O}_{X_k}(k') \otimes \pi_{0,k}^* A^{(k+1)(\epsilon_p + k\delta_p)} \otimes \mathfrak{w}_\infty(X_k)),$$

and thus there exists

$$\hat{\omega}_{I_0, \dots, I_k}(a_{I_0}, \dots, a_{I_k}) \in H^0\left(\hat{X}_k, \nu_k^*(\mathcal{O}_{X_k}(k') \otimes \pi_{0,k}^* A^{(k+1)(\epsilon_p + k\delta_p)}) \otimes \mathcal{O}_{\hat{X}_k}(-F)\right),$$

such that

$$\nu_k^* \omega_{I_0, \dots, I_k}(a_{I_0}, \dots, a_{I_k}) = s_F \cdot \hat{\omega}_{I_0, \dots, I_k}(a_{I_0}, \dots, a_{I_k}).$$

Therefore, if we define the rational map

$$\begin{aligned} \Phi^p : S_p \times \hat{X}_k &\dashrightarrow P(\Lambda^{k+1} \mathbb{C}^{\mathbb{I}^p}) \\ (\mathbf{a}, \hat{w}) &\mapsto ([\hat{\omega}_{I_0, \dots, I_k}(a_{I_0}, \dots, a_{I_k})(\hat{w})])_{I_0, \dots, I_k \in \mathbb{I}^p}, \end{aligned}$$

then on  $\hat{X}_k \setminus F$  we have  $\hat{\Phi}^p = \Phi^p \circ \nu_k$ , and thus  $\hat{\Phi}^p$  also factors through the Plücker embedding

$$\text{Pluc} : \text{Gr}_{k+1}(\mathbb{C}^{\mathbb{I}^p}) \hookrightarrow P(\Lambda^{k+1} \mathbb{C}^{\mathbb{I}^p}).$$

We are going to show that  $\nu_k$  partially resolves the indeterminacy of  $\Phi^p$ . To clarify this, we need to introduce some notations. For any  $x \in X$ , we set

$$N_x := \#\{j \in \{0, \dots, N\} \mid \tau_j(x) \neq 0\} \text{ and } \mathbb{I}_x^p := \{I \in \mathbb{I}^p, |\tau^I(x)| \neq 0\}.$$

Since the  $\tau_j$ 's are in general position, and  $N \geq n$ , we have  $N_x \geq 1$  for all  $x \in X$ . Then we define

$$\Sigma := \{x \in X \mid N_x = 1\} \text{ and } X^\circ := X \setminus \Sigma.$$

Observe that if  $N > n$ , then  $X^\circ = X$ , and if  $N = n$ , then  $\Sigma$  is a finite set of points. We denote by  $\hat{X}_k^\circ := (\pi_{0,k} \circ \nu_k)^{-1}(X^\circ)$ . We have the following crucial lemma of resolution of indeterminacy due to Brotbek [Bro16, Proposition 3.8]:

**Lemma 3.3.** (*Brotbek*) Suppose that

$$N \geq n \geq 2, \quad k \geq 1, \quad \epsilon_p \geq m_\infty(X_k, A) = k \text{ and } \delta_p \geq n(k+1). \quad (\star)$$

Then there exists a non-empty Zariski open subset  $U_{\text{def},p} \subset S_p$  such that the restriction  $\hat{\Phi}^p|_{U_{\text{def},p} \times \hat{X}_k^\circ}$  is a morphism:

$$\begin{array}{ccc} U_{\text{def},p} \times \hat{X}_k^\circ & & \\ \mathbf{1} \times \nu_k \downarrow & \searrow \hat{\Phi}^p & \\ U_{\text{def},p} \times X_k^\circ & \xrightarrow{\hat{\Phi}^p} & \text{Gr}_{k+1}(\mathbb{C}^{\mathbb{I}^p}) \end{array}$$

In Lemma 3.3, we have applied our Theorem 2.4 to set  $m_\infty(X_k, A) = k$ .

### 3.2 MAPPING TO THE UNIVERSAL GRASSMANNIAN

Set  $U_{\text{def}} := U_{\text{def},1} \times \dots \times U_{\text{def},c} \cap S$ . We suppose from now on that  $N \geq n \geq 2$ ,  $\epsilon_p \geq k \geq 1$  and that  $\delta_p \geq n(k+1)$  for any  $1 \leq p \leq c$ . Then by Lemma 3.3 we get a regular morphism

$$\begin{aligned} \Psi : U_{\text{def}} \times \hat{X}_k^\circ &\rightarrow \mathbf{G}_{k+1}(\delta_1) \times \dots \times \mathbf{G}_{k+1}(\delta_c) \times \mathbb{P}^N \\ (\mathbf{a}, \xi) &\mapsto (\hat{\Phi}^1(\mathbf{a}^1, \xi), \dots, \hat{\Phi}^c(\mathbf{a}^c, \xi), [\tau^r(\xi)]). \end{aligned}$$

$[\tau^r(\xi)] := [\tau_0^r(\pi_{0,k} \circ \nu_k(\xi)) : \dots : \tau_N^r(\pi_{0,k} \circ \nu_k(\xi))]$ , and we write  $\mathbf{G}_{k+1}(\delta_p) := \text{Gr}_{k+1}(\mathbb{C}^{\mathbb{I}^p})$  and  $\mathbf{G} := \mathbf{G}_{k+1}(\delta_1) \times \dots \times \mathbf{G}_{k+1}(\delta_c)$  for brevity.

From now on we *always* assume that  $(k+1)c \geq N$ . Using the natural identification

$$\begin{aligned} \mathbb{C}^{\mathbb{I}^p} &\rightarrow H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(\delta_i)) \\ (a_I)_{I \in \mathbb{I}^p} &\mapsto \sum_{I \in \mathbb{I}^p} a_I z^I, \end{aligned}$$

we set  $\mathcal{Y}$  to be the *universal Grassmannian* defined by

$$\mathcal{Y} := \{(\Delta_1, \dots, \Delta_c, [z]) \in \mathbf{G} \times \mathbb{P}^N \mid \forall 1 \leq i \leq c, \forall P \in \Delta_i : P([z]) = 0\}.$$

If we denote by  $p : \mathcal{Y} \rightarrow \mathbf{G}$  the first projection map, then  $p$  is a generically finite to one (may not surjective) morphism. Set  $G^\infty$  to be the set of points in  $\mathbf{G} := \mathbf{G}_{k+1}(\delta_1) \times \dots \times \mathbf{G}_{k+1}(\delta_c)$  such that the fiber of  $p : \mathcal{Y} \rightarrow \mathbf{G}$  is not a finite set, and we say that  $G^\infty$  is the *non-finite loci* of  $\mathbf{G}$ .

We need to cover  $X$  by a natural stratification induced by the vanishing of the  $\tau_j$ 's. For any  $J \subset \{0, \dots, N\}$  and  $1 \leq p \leq c$  we define

$$\begin{aligned} X_J &:= \{x \in X \mid \tau_j(x) = 0 \Leftrightarrow j \in J\}, \\ \mathbb{I}_J^p &:= \{I \in \mathbb{I}^p \mid \text{Supp}(I) \subset \{0, \dots, N\} \setminus J\}, \\ \mathbb{P}_J &:= \{[z] \in \mathbb{P}^N \mid z_j = 0 \text{ iff } j \in J\}, \end{aligned}$$

$\hat{X}_{k,J} := (\pi_{0,k} \circ \nu_k)^{-1}(X_J)$  and  $\hat{X}_{k,J}^\circ := \hat{X}_{k,J} \cap \hat{X}_k^\circ$ . Set  $\mathcal{Y}_J := \mathcal{Y} \cap (\mathbf{G} \times \mathbb{P}_J) \subset \mathbf{G} \times \mathbb{P}^N$ , and  $G_J^\infty$  also the set of points in  $\mathbf{G}$  such that the fiber of the first projection map  $p_J : \mathcal{Y}_J \rightarrow \mathbf{G}$  is not a finite set.

Now set

$$U_{\text{def},p}^\circ := U_{\text{def},p} \cap \{\mathbf{a}^p \in S_p \mid \{\mathbf{F}^p(\mathbf{a}^p)(x) = 0\} \cap \Sigma = \emptyset\} \text{ and } U_{\text{def}}^\circ := U_{\text{def},1}^\circ \times \dots \times U_{\text{def},c}^\circ \cap U_{\text{def}}.$$

Since  $\Sigma$  is a finite set,  $U_{\text{def},p}^\circ$  is a non-empty Zariski open subset of  $U_{\text{def},p}$  for each  $p$ . Consider the universal family of codimension  $c$  smooth varieties  $\mathcal{H} := (U_{\text{def}}^\circ \times X) \cap \mathcal{X}$ , then

$$\mathcal{H} \cap \{U_{\text{def}}^\circ \times \Sigma\} = \emptyset. \quad (3.7)$$

We denote by  $\mathcal{H}_k^{\text{rel}}$  the  $k$ -th Demailly-Semple tower of the relative directed variety  $(\mathcal{H}, T_{\mathcal{H}/U_{\text{def}}^\circ})$ . If  $\hat{\mathcal{H}}_k^{\text{rel}}$  is obtained by the blowing-up of the asymptotic Wronskian ideal sheaf  $\mathfrak{w}_\infty(\mathcal{H}_k^{\text{rel}})$ , then by the arguments in Section 2.4 we have

$$(\mathbb{1} \times \nu_k)^{-1}(\mathcal{H}_k^{\text{rel}}) = \hat{\mathcal{H}}_k^{\text{rel}}.$$

Moreover for any  $\mathbf{a} \in U_{\text{def}}^\circ$ , if we denote by  $H_{\mathbf{a},k} := \mathcal{H}_k^{\text{rel}} \cap (\{\mathbf{a}\} \times X_k)$  and  $\hat{H}_{\mathbf{a},k} := \hat{\mathcal{H}}_k^{\text{rel}} \cap (\{\mathbf{a}\} \times \hat{X}_k)$ , then  $\nu_k|_{\hat{H}_{\mathbf{a},k}} : \hat{H}_{\mathbf{a},k} \rightarrow H_{\mathbf{a},k}$  is indeed the blowing-up of the asymptotic Wronskian ideal sheaf  $\mathfrak{w}_\infty(H_{\mathbf{a},k})$ . By (3.7),  $\Psi|_{\hat{\mathcal{H}}_k^{\text{rel}}}$  is a regular morphism. Set

$$\hat{\mathcal{H}}_{k,J}^{\text{rel}} := \hat{\mathcal{H}}_k^{\text{rel}} \cap (U_{\text{def}}^\circ \times \hat{X}_{k,J}),$$

and we have the following

**Proposition 3.1.** *For any  $J \subset \{0, \dots, N\}$ , when restricted to  $\hat{\mathcal{H}}_{k,J}^{\text{rel}}$  the morphism  $\Psi$  factors through  $\mathcal{Y}_J$ :*

$$\Psi|_{\hat{\mathcal{H}}_{k,J}^{\text{rel}}} : \hat{\mathcal{H}}_{k,J}^{\text{rel}} \rightarrow \mathcal{Y}_J \subset \mathbf{G} \times \mathbb{P}_J.$$

*Proof.* Since when restricted to  $U_{\text{def}}^\circ \times \hat{X}_{k,J}^\circ$ ,  $\Psi$  factors through  $\mathbf{G} \times \mathbb{P}_J$ . Thus it suffices to prove that  $\Psi|_{\hat{\mathcal{H}}_k^{\text{rel}}}$  factors through  $\mathcal{Y}$ . Since  $\Phi^p = \hat{\Phi}^p \circ \nu_k$ , it suffices to prove that the rational map

$$\begin{aligned} \tilde{\Psi} : S \times X_k &\dashrightarrow \mathbf{G} \times \mathbb{P}^N \\ (\mathbf{a}, w) &\mapsto (\Phi^1(\mathbf{a}^1, w), \dots, \Phi^c(\mathbf{a}^c, w), [\tau^r(w)]) \end{aligned}$$

factor through  $\mathcal{Y}$  when restricted to  $\hat{\mathcal{H}}_k^{\text{rel}}$ . Take any  $(\mathbf{a}, w_0) \in \hat{\mathcal{H}}_k^{\text{rel}}$  outside the indeterminacy of  $\tilde{\Psi}$ , and by Lemma 2.1 one can find a germ of curve  $f : (\mathbb{C}, 0) \rightarrow (X, x := \pi_{0,k}(w_0))$  with  $f_{[k]}(0) = w_0$ . Recall that  $H_{\mathbf{a},k} := \mathcal{H}_k^{\text{rel}} \cap (\{\mathbf{a}\} \times X_k)$  is the  $k$ -th Demailly-Semple tower of  $(H_{\mathbf{a}}, T_{H_{\mathbf{a}}})$ . Therefore, we have  $(f(0), f'(0), \dots, f^{(k)}(0)) \in J_k H_{\mathbf{a}}$ .

Take an open subset  $U \subset X$  containing  $x$  such that  $A|_U$  can be trivialized. Since  $H_{\mathbf{a}}$  is defined by the equations

$$\begin{cases} \mathbf{F}^1(\mathbf{a}^1)(x) := \sum_{|I|=\delta_1} a_I^1(x) \tau(x)^{(r+k)I} = 0, \\ \vdots \\ \mathbf{F}^c(\mathbf{a}^c)(x) := \sum_{|I|=\delta_c} a_I^c(x) \tau(x)^{(r+k)I} = 0, \end{cases}$$

then  $d_U^{[j]} \mathbf{F}^p(\mathbf{a}^p)(f', f'', \dots, f^{(k)}) = 0$  for any  $1 \leq p \leq c$  and  $0 \leq j \leq k$ . By Lemma 3.1 we have  $d_U^{[j]} \mathbf{F}^p(\mathbf{a}^p) = \tau_U^{rI} \cdot \sum_{|I|=\delta_p} d_{I,U}^{[j]}(a_I^p)$ . By the definition for  $\Phi^p$  (3.6), we see that  $\tilde{\Psi}(\mathbf{a}, w_0) \in \mathcal{Y}$ . This completes the proof of the Proposition.  $\square$

To proceed further, we need another important technical lemma in [Bro16, Lemma 3.4] as follows

**Lemma 3.4.** *Suppose that  $\epsilon \geq m_\infty(X_k, A) = k$ . Fix any  $1 \leq p \leq c$ . For any  $\hat{w}_0 \in \hat{X}_k$ , there exists an open neighborhood  $\hat{U}_{\hat{w}_0} \subset \hat{X}_k$  of  $\hat{w}_0$  satisfying the following. For any  $I \in \mathbb{I}^p$  and  $0 \leq i \leq k$  there exists a linear map*

$$g_{i,I}^p : H^0(X, A^\epsilon) \rightarrow \mathcal{O}(\hat{U}_{\hat{w}_0})$$

*such that for any  $(\mathbf{a}^p, \hat{w}) \in S_p \times \hat{U}_{\hat{w}_0}$ , writting  $g_{i,\bullet}^p(\mathbf{a}^p, \hat{w}) = (g_{i,I}^p(a_I^p)(\hat{w}))_{I \in \mathbb{I}^p} \in \mathbb{C}^{\mathbb{I}^p}$  one has*

(i) *The Plücker coordinates of  $\hat{\Phi}^p(\mathbf{a}^p, \hat{w})$  are all vanishing if and only if*

$$\dim \text{Span}(g_{0,\bullet}^p(\mathbf{a}^p, \hat{w}), \dots, g_{k,\bullet}^p(\mathbf{a}^p, \hat{w})) < k + 1.$$

(ii) *If  $\dim \text{Span}(g_{0,\bullet}^p(\mathbf{a}^p, \hat{w}), \dots, g_{k,\bullet}^p(\mathbf{a}^p, \hat{w})) = k + 1$ , then*

$$\hat{\Phi}^p(\mathbf{a}^p, \hat{w}) = \text{Span}(g_{0,\bullet}^p(\mathbf{a}^p, \hat{w}), \dots, g_{k,\bullet}^p(\mathbf{a}^p, \hat{w})) \in \text{Gr}_{k+1}(\mathbb{C}^{\mathbb{I}^p}).$$

(iii) *Define the linear map*

$$\begin{aligned} \hat{\varphi}_{\hat{w}_0}^p : S_p &\rightarrow (\mathbb{C}^{\mathbb{I}^p})^{k+1} \\ \mathbf{a}^p &\mapsto (g_{0,\bullet}^p(\mathbf{a}^p, \hat{w}_0), \dots, g_{k,\bullet}^p(\mathbf{a}^p, \hat{w}_0)). \end{aligned} \tag{3.8}$$

*Set  $x := \pi_{0,k} \circ \nu_k(\hat{w}_0)$  and  $\rho_x^p : (\mathbb{C}^{\mathbb{I}^p})^{k+1} \rightarrow (\mathbb{C}^{\mathbb{I}_x^p})^{k+1}$  the natural projection map, then one has*

$$\text{rank} \rho_x^p \circ \hat{\varphi}_{\hat{w}_0}^p = (k+1) \# \mathbb{I}_x^p.$$

*Here  $\mathbb{I}_x^p := \{I \in \mathbb{I}^p | \tau^I(x) \neq 0\}$ .*

Now we are ready to prove the following lemma, which is a variant of [Bro16, Lemma 3.9]:

**Lemma 3.5.** *(Avoiding exceptional locus) For any  $J \subset \{0, \dots, N\}$ . If  $\delta_p \geq (n-1)(k+1) + 1$  for any  $p = 1, \dots, c$ , then there exists a non-empty Zariski open subset  $U_J \subset U_{\text{def}}^\circ$  such that*

$$\hat{\Phi}^{-1}(G_J^\infty) \cap (U_J \times \hat{X}_{k,J}^\circ) = \emptyset.$$

*Here we define the map (which is a morphism by Lemma 3.3)*

$$\begin{aligned} \hat{\Phi} : U_{\text{def}}^\circ \times \hat{X}_k^\circ &\rightarrow \mathbf{G}_{k+1}(\delta_1) \times \dots \times \mathbf{G}_{k+1}(\delta_c) \\ (\mathbf{a}, \xi) &\mapsto (\hat{\Phi}^1(\mathbf{a}^1, \xi), \dots, \hat{\Phi}^c(\mathbf{a}^c, \xi)), \end{aligned}$$

*which is the composition  $\pi_1 \circ \Psi$ . Here  $\pi_1 : \mathbf{G} \times \mathbb{P}^N \rightarrow \mathbf{G}$  is the first projection.*

*Proof.* Fix any  $\hat{w}_0 \in \hat{X}_k^\circ$ , we set  $x := \pi_{0,k} \circ \nu_k(\hat{w}_0)$ . Then there exists a unique  $J \subset \{0, \dots, N\}$  such that  $x \in X_J$ , and we define the following analogues of  $\mathcal{V}$  parametrized by affine spaces

$$\begin{aligned}\widetilde{\mathcal{V}} &:= \{(\alpha_{10}, \dots, \alpha_{1k}, \dots, \alpha_{c0}, \dots, \alpha_{ck}, [z]) \in \prod_{p=1}^c (\mathbb{C}^{\mathbb{I}^p})^{k+1} \times \mathbb{P}_J \mid \forall 1 \leq p \leq c, 0 \leq j \leq k, \alpha_{pi}([z]) = 0\}, \\ \widetilde{\mathcal{V}}_x &:= \{(\alpha_{10}, \dots, \alpha_{1k}, \dots, \alpha_{c0}, \dots, \alpha_{ck}, [z]) \in \prod_{p=1}^c (\mathbb{C}^{\mathbb{I}_J^p})^{k+1} \times \mathbb{P}_J \mid \forall 1 \leq p \leq c, 0 \leq j \leq k, \alpha_{pi}([z]) = 0\},\end{aligned}$$

here we use the identification  $\mathbb{C}^{\mathbb{I}^p} \cong H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(\delta_p))$  and  $\mathbb{C}^{\mathbb{I}_J^p} \cong H^0(\mathbb{P}_J, \mathcal{O}_{\mathbb{P}_J}(\delta_p))$ . By analogy with  $G_J^\infty$ , we denote by  $V_{1,J}^\infty$  (resp.  $V_{2,J}^\infty$ ) the set of points in  $\prod_{p=1}^c (\mathbb{C}^{\mathbb{I}^p})^{k+1}$  (resp.  $\prod_{p=1}^c (\mathbb{C}^{\mathbb{I}_J^p})^{k+1}$ ) at which the fiber in  $\widetilde{\mathcal{V}}$  (resp.  $\widetilde{\mathcal{V}}_x$ ) is positive dimensional.

For every  $1 \leq p \leq c$  we take the linear map  $\hat{\varphi}_{\hat{w}_0}^p : S_p \rightarrow (\mathbb{C}^{\mathbb{I}^p})^{k+1}$  as in (3.8). By Lemma 3.4, for any  $\mathbf{a} \in U_{\text{def}}^\circ$  we have

$$\hat{\Phi}(\mathbf{a}, \hat{w}_0) = ([\hat{\varphi}_{\hat{w}_0}^1(\mathbf{a}^1)], \dots, [\hat{\varphi}_{\hat{w}_0}^c(\mathbf{a}^c)]),$$

here  $[\hat{\varphi}_{\hat{w}_0}^p(\mathbf{a}^p)] := \text{Span}(g_{0,\bullet}^p(\mathbf{a}^p, \hat{w}_0), \dots, g_{k,\bullet}^p(\mathbf{a}^p, \hat{w}_0)) \in \text{Gr}_{k+1}(\mathbb{C}^{\mathbb{I}^p})$ . Then we have

$$\hat{\Phi}^{-1}(G_J^\infty) \cap (U_{\text{def}}^\circ \times \{\hat{w}_0\}) = \hat{\varphi}_{\hat{w}_0}^{-1}(V_{1,J}^\infty) \cap U_{\text{def}}^\circ = (\rho_x \circ \hat{\varphi}_{\hat{w}_0})^{-1}(V_{2,J}^\infty) \cap U_{\text{def}}^\circ,$$

where we denote by

$$\begin{aligned}\hat{\varphi}_{\hat{w}_0} : S_1 \times \dots \times S_c &\rightarrow \prod_{p=1}^c (\mathbb{C}^{\mathbb{I}^p})^{k+1} \\ \mathbf{a} = (\mathbf{a}^1, \dots, \mathbf{a}^c) &\mapsto (\hat{\varphi}_{\hat{w}_0}^1(\mathbf{a}^1), \dots, \hat{\varphi}_{\hat{w}_0}^c(\mathbf{a}^c)),\end{aligned}$$

and

$$\rho_x : \prod_{p=1}^c (\mathbb{C}^{\mathbb{I}^p})^{k+1} \rightarrow \prod_{p=1}^c (\mathbb{C}^{\mathbb{I}_x^p})^{k+1}$$

is the natural projection map. By the above notations we have  $\mathbb{I}_J^p = \mathbb{I}_x^p$  for any  $p = 1, \dots, c$ . Since the linear map  $\rho_x \circ \hat{\varphi}_{\hat{w}_0}$  is diagonal by blocks, by Lemma 3.4 we have

$$\text{rank} \rho_x \circ \hat{\varphi}_{\hat{w}_0} = \sum_{p=1}^c \text{rank} \rho_x^p \circ \hat{\varphi}_{\hat{w}_0}^p = \sum_{p=1}^c (k+1) \# \mathbb{I}_x^p.$$

Therefore

$$\begin{aligned}\dim(\hat{\Phi}^{-1}(G_J^\infty) \cap (U_{\text{def}}^\circ \times \{\hat{w}_0\})) &\leq \dim((\rho_x \circ \hat{\varphi}_{\hat{w}_0})^{-1}(V_{2,J}^\infty)) \\ &\leq \dim(V_{2,J}^\infty) + \dim \ker(\rho_x \circ \hat{\varphi}_{\hat{w}_0}) \\ &\leq \dim(V_{2,J}^\infty) + \dim(S_1 \times \dots \times S_c) - \text{rank}(\rho_x \circ \hat{\varphi}_{\hat{w}_0}) \\ &= \dim(V_{2,J}^\infty) + \dim(S_1 \times \dots \times S_c) - \sum_{p=1}^c (k+1) \# \mathbb{I}_x^p.\end{aligned}$$

Since

$$\begin{aligned}\dim(V_{2,J}^\infty) &= \dim\left(\prod_{p=1}^c (\mathbb{C}^{\mathbb{I}_J^p})^{k+1}\right) - \text{codim}(V_{2,J}^\infty, \prod_{p=1}^c (\mathbb{C}^{\mathbb{I}_J^p})^{k+1}) \\ &= \sum_{p=1}^c (k+1) \# \mathbb{I}_J^p - \text{codim}(V_{2,J}^\infty, \prod_{p=1}^c (\mathbb{C}^{\mathbb{I}_J^p})^{k+1}) \\ &= \sum_{p=1}^c (k+1) \# \mathbb{I}_x^p - \text{codim}(V_{2,J}^\infty, \prod_{p=1}^c (\mathbb{C}^{\mathbb{I}_J^p})^{k+1}),\end{aligned}$$

then we have

$$\dim(\hat{\Phi}^{-1}(G_J^\infty) \cap U_{\text{def}}^\circ \times \{\hat{w}_0\}) \leq \dim(S_1 \times \dots \times S_c) - \text{codim}(V_{2,J}^\infty, \prod_{p=1}^c (\mathbb{C}^{\mathbb{P}_J^p})^{k+1}),$$

which yields

$$\dim(\hat{\Phi}^{-1}(G_J^\infty) \cap U_{\text{def}}^\circ \times \hat{X}_{k,J}^\circ) \leq \dim(S_1 \times \dots \times S_c) - \text{codim}(V_{2,J}^\infty, \prod_{p=1}^c (\mathbb{C}^{\mathbb{P}_J^p})^{k+1}) + \dim \hat{X}_k.$$

By a result due to Olivier Benoist [BD15, Corollary 3.2], we have

$$\text{codim}(V_{2,J}^\infty, \prod_{p=1}^c (\mathbb{C}^{\mathbb{P}_J^p})^{k+1}) \geq \min_{1 \leq p \leq c} \delta_p + 1.$$

Therefore, if

$$\dim \hat{X}_k < \min_{1 \leq p \leq c} \delta_p + 1, \quad (\clubsuit)$$

$\hat{\Phi}^{-1}(G_J^\infty)$  doesn't dominate  $U_{\text{def}}^\circ$  via the projection  $U_{\text{def}}^\circ \times \hat{X}_{k,J}^\circ \rightarrow U_{\text{def}}^\circ$ , and thus we can find a non-empty Zariski open subset  $U_J \subset U_{\text{def}}^\circ$  such that

$$\hat{\Phi}^{-1}(G_J^\infty) \cap (U_J \times \hat{X}_{k,J}^\circ) = \emptyset.$$

Thus if  $\min_{1 \leq p \leq c} \delta_p \geq (n-1)(k+1) + 1$ , Condition  $\clubsuit$  is always satisfied. We finish the proof of the lemma.  $\square$

### 3.3 PULL-BACK OF THE POSITIVITY

For any  $c$ -tuple of positive integers  $\mathbf{e} = (e_1, \dots, e_c)$ , we denote by

$$\mathcal{L}(\mathbf{e}) := \mathcal{O}_{\mathbf{G}_{k+1}(\delta_1)}(e_1) \boxtimes \dots \boxtimes \mathcal{O}_{\mathbf{G}_{k+1}(\delta_c)}(e_c),$$

which is a very ample line bundle on  $\mathbf{G}$ . Since  $p_J : \mathcal{Y}_J \rightarrow \mathbf{G}$  is a generically finite to one morphism, by the Nakamaye Theorem (see [Laz04, Theorem 10.3.5] for smooth projective varieties, and [Bir13, Theorem 1.3] for any projective scheme over any field), the augmented base locus  $\mathbb{B}_+(p_J^* \mathcal{L}(\mathbf{e}))$  for  $p_J^* \mathcal{L}(\mathbf{e})$  coincides with its *exceptional locus* (or say *null locus*)

$$E_J := \{y \in \mathcal{Y} \mid \dim_y(p_J^{-1}(p_J(y))) > 0\},$$

which is contained in  $p_J^{-1}(G_J^\infty)$ . Thus if  $e_i \gg 0$  for each  $1 \leq i \leq c$ , we have

$$E_J = \text{Bs}(p_J^* \mathcal{L}(\mathbf{e}) \otimes q_J^* \mathcal{O}_{P_J}(-1)) \subset p_J^{-1}(G_J^\infty), \quad (3.9)$$

where  $q_J : \mathcal{Y}_J \rightarrow \mathbb{P}_J$  is denoted to be the second projection map. In Section 4, we obtain an effective estimate for  $\mathbf{e}$  such that the inclusive relation in (3.9) holds. The theorem is the following

**Theorem 3.1.** *With the above notations, set  $b_p := \frac{\prod_{j=1}^c \delta_j^{k+1}}{\delta_p}$ , then for any  $J \subset \{0, \dots, N\}$  and any  $\mathbf{a} \in \mathbb{Z}^c$  with  $a_p \geq b_p$  for every  $1 \leq p \leq c$ , we have*

$$\text{Bs}(p_J^* \mathcal{L}(\mathbf{a}) \otimes q_J^* \mathcal{O}_{\mathbb{P}^N}(-1)) \subset \text{Bs}(p_J^* \mathcal{L}(\mathbf{b}) \otimes q_J^* \mathcal{O}_{\mathbb{P}^N}(-1)) \subset p_J^{-1}(G_J^\infty).$$

Since the technique in proving this theorem is of independent interest, we will leave the proof to Section 4.

**Remark 3.1.** *Since  $p_J^{-1}(G_J^\infty)$  may strictly contain the null locus  $\text{Null}(p^* \mathcal{L}|_{\mathcal{Y}_J}) = E_J$ , Theorem 3.1 does not imply the Nakamaye Theorem used in [BD15] and [Bro16]. That is, for some  $J$  and  $\mathbf{a} \in \mathbb{N}^c$  with  $a_j \geq b_j$  for every  $j$ , the Null locus  $E_J$  may be strictly contained in  $p_J^* \mathcal{L}(\mathbf{a}) \otimes q_J^* \mathcal{O}_{\mathbb{P}^N}(-1)$ . However, as we will see later, our proof of the Main Theorem only relies on the inclusive relation in (3.9). We thank Brotbek for pointing this important reduction to us.*



By (3.4) we have

$$\Psi^*(\mathcal{L}(\mathbf{b}) \boxtimes \mathcal{O}_{\mathbb{P}^N}(-1)) = \nu_k^*(\mathcal{O}_{X_k}(\sum_{p=1}^c b_p k') \otimes \pi_{0,k}^* A^{-q(\epsilon, \delta, r)}) \otimes \mathcal{O}_{\hat{X}_k}(-\sum_{p=1}^c b_p F). \quad (3.10)$$

Here we set  $q(\epsilon, \delta, r) := r - \sum_{p=1}^c b_p(k+1)(\epsilon_p + k\delta_p)$ . Observe that if we take

$$\sum_{p=1}^c b_p(k+1)(\epsilon_p + k\delta_p) < r, \quad (\spadesuit)$$

then (3.10) becomes an *invariant*  $k$ -jet differential with a negative twist, which enables us to apply Theorem 2.3 to constrain all the entire curves. More precisely, we have the following theorem:

**Theorem 3.2.** *On an  $n$ -dimensional smooth projective variety  $X$ , equipped with a very ample line bundle  $A$ . Let  $c$  be any integer satisfying  $1 \leq c \leq n-1$ . If we take  $k_0 = \lceil \frac{n}{c} \rceil - 1$  and  $N = n$ , then for any degrees  $(d_1, \dots, d_c) \in (\mathbb{N})^c$  satisfying*

$$\begin{aligned} \exists \delta_{(\delta_p \geq \delta_0 := n(k_0+1))}, \exists \epsilon_{(\epsilon_p \geq k_0)}, \exists r > \sum_{p=1}^c b_p(k_0+1)(\epsilon_p + k_0\delta_p), \text{ s.t.} \\ d_p = \delta_p(r + k_0) + \epsilon_p \quad (p = 1, \dots, c), \end{aligned} \quad (3.11)$$

the complete intersection  $\mathbf{H} := H_1 \cap \dots \cap H_c$  of general hypersurfaces  $H_1 \in |A^{d_1}|, \dots, H_c \in |A^{d_c}|$  has almost  $k$ -jet ampleness.

*Proof.* We will prove the theorem in several steps. First observe that, the choice for  $(\epsilon, \delta, r, c, N, k)$  in the Theorem fulfills all the requirements in Condition  $\star$ ,  $\spadesuit$  and  $\clubsuit$ , and thus we are free to apply all the corresponding theorems above. Based on the same vein in [BD15, Bro16], we have the following result

**Claim 3.3.** *Set  $U_{\text{nef}} := \cap_J U_J$ . For any  $\mathbf{a} \in U_{\text{nef}}$ , the line bundle*

$$\nu_k^*(\mathcal{O}_{X_k}(\sum_{p=1}^c b_p k') \otimes \pi_{0,k}^* A^{-q(\epsilon, \delta, r)}) \otimes \mathcal{O}_{\hat{X}_k}(-\sum_{p=1}^c b_p F)|_{\hat{H}_{\mathbf{a},k}}$$

is nef on  $\hat{H}_{\mathbf{a},k}$ . Recall that we denote by  $q(\epsilon, \delta, r) := r - \sum_{p=1}^c b_p(k_0+1)(\epsilon_p + k_0\delta_p) > 0$ .

**Proof:** In order to prove nefness, it suffices to show that for any irreducible curve, its intersection with the line bundle is non-negative. For any fixed  $\mathbf{a} \in U_{\text{nef}}$ , and any irreducible curve  $C \subset \hat{H}_{\mathbf{a},k}$ , one can find the unique  $J \subset \{0, \dots, N\}$  such that  $\hat{X}_{k,J} \cap C =: C^\circ$  is a non-empty Zariski open subset of  $C$ , and thus  $C^\circ \subset \hat{\mathcal{H}}_{k,J}$ . From Proposition 3.1,  $\Psi$  factors through  $\mathcal{Y}_J$  when restricted to  $\hat{\mathcal{H}}_{k,J}$ , thus  $\Psi|_{C^\circ}$  also factors through  $\mathcal{Y}_J$ , and by the properness of  $\mathcal{Y}_J$ ,  $\Psi|_C$  factors through  $\mathcal{Y}_J$  as well. By Lemma 3.5 and the definition of  $U_{\text{nef}}$ , we have

$$\hat{\Phi}(C^\circ) \cap G_J^\infty = \emptyset,$$

and thus

$$\Psi(C) \not\subset p_J^{-1}(G_J^\infty).$$

From Theorem 3.1 we know that

$$\text{Bs}(p_J^* \mathcal{L}(\mathbf{b}) \otimes q_J^* \mathcal{O}_{P_J}(-1)) \subset p_J^{-1}(G_J^\infty),$$

which yields

$$\Psi(C) \cdot (p_J^* \mathcal{L}(\mathbf{b}) \otimes q_J^* \mathcal{O}_{P_J}(-1)) \geq 0.$$

From the relation (3.10) we obtain that

$$C \cdot \left( \nu_k^*(\mathcal{O}_{X_k}(\sum_{p=1}^c b_p k') \otimes \pi_{0,k}^* A^{-q(\epsilon, \delta, r)}) \otimes \mathcal{O}_{\hat{X}_k}(-\sum_{p=1}^c b_p F) \right) \geq 0,$$

which proves the claim. ■

By [Dem95, Proposition 6.16], we can find an ample line bundle on  $\hat{X}^k$  of the form

$$\tilde{A} := \nu_k^*(\mathcal{O}_{X_k}(a_1, \dots, a_k) \otimes \pi_{0,k}^* A^{a_0}) \otimes \mathcal{O}_{\hat{X}_k}(-F)$$

for some  $a_0, \dots, a_k \in \mathbb{N}$ . Therefore, for any  $m > a_0$ , the line bundle

$$\nu_k^*(\mathcal{O}_{X_k}(a_1, \dots, a_{k-1}, a_k + \sum_{p=1}^c mb_p k') \otimes \pi_{0,k}^* A^{a_0 - mq(\epsilon, \delta, r)}) \otimes \mathcal{O}_{\hat{X}_k}(-(\sum_{p=1}^c mb_p + 1)F)|_{\hat{H}_{\mathbf{a},k}}$$

is ample for any  $\mathbf{a} \in U_{\text{nef}}$ , and thus there exists  $e_0, \dots, e_k \in \mathbb{N}$  such that

$$\nu_k^* \mathcal{O}_{X_k}(e_1, \dots, e_k) \otimes \mathcal{O}_{\hat{X}_k}(-e_0 F)|_{\hat{H}_{\mathbf{a},k}}$$

is ample. By the openness property of ampleness, one has a non-empty Zariski open subset  $U_{\text{ample}} \subset \prod_{i=1}^c |A^{d_i}|$  such that for any  $(H_1, \dots, H_c) \in U_{\text{ample}}$ , their intersection  $\mathbf{H} := H_1 \cap \dots \cap H_c$  is a reduced smooth variety of codimension  $c$  in  $X$ , and the restriction of the line bundle  $\nu_k^* \mathcal{O}_{X_k}(e_1, \dots, e_k) \otimes \mathcal{O}_{\hat{X}_k}(-qF)|_{\hat{\mathbf{H}}_k}$  is ample (recall that  $\hat{\mathbf{H}}_k$  is denoted to be the blow-up of  $\mathbf{H}_k$  along  $\mathfrak{w}_\infty(\mathbf{H}_k)$ ). Since the exceptional locus of the blow-up  $\nu_k : \hat{X}_k \rightarrow X_k$  is contained in  $X_k^{\text{sing}}$ , then for the complete intersection  $\mathbf{H} := H_1 \cap \dots \cap H_c$  of general hypersurfaces  $H_1 \in |A^{d_1}|, \dots, H_c \in |A^{d_c}|$ , the augmented base locus of the line bundle

$$\mathcal{O}_{\mathbf{H}_k}(e_1, \dots, e_k) = \mathcal{O}_{X_k}(e_1, \dots, e_k)|_{\mathbf{H}_k}$$

is contained in  $X_k^{\text{sing}} \cap \mathbf{H}_k$ , and we conclude that  $H_k$  has almost  $k$ -jet ampleness by the fact that  $X_k^{\text{sing}} \cap \mathbf{H}_k = \mathbf{H}_k^{\text{sing}}$ . □

Now we make some effective estimates based on Theorem 3.2. If we take

$$d_0 := \delta_0(c(k_0 + 1)(k_0 + \delta_0 + k_0\delta_0 - 1)\delta_0^{c(k_0+1)-1} + k_0 + 1) + k_0,$$

then any  $d \geq d_0$  has a decomposition

$$d = (t + k_0)\delta_0 + \epsilon$$

with  $k_0 \leq \epsilon < \delta_0 + k_0$  and  $t \geq c(k_0 + 1)(k_0 + \delta_0 + k_0\delta_0 - 1)\delta_0^{c(k_0+1)-1} + 1$ , satisfying the conditions in Theorem 3.2. Therefore, the complete intersection  $H_1 \cap \dots \cap H_c$  of general hypersurfaces  $H_1, \dots, H_c \in |A^d|$  with  $d \geq d_0$  has almost  $k_0$ -jet ampleness. By [Dem95, Lemma 7.6], if a complex manifold  $Y$  has almost  $k$ -jet ampleness, then it will also have almost  $l$ -jet ampleness for any  $l \geq k$ . A computation gives a rough estimate  $d_0 \leq 2c(\lceil \frac{n}{c} \rceil)^{n+c+2}n^{n+c}$ , and this completes the proof of Theorem A.

### 3.4 UNIFORM ESTIMATES FOR THE LOWER BOUNDS ON THE DEGREE

In Theorem 3.2, the lower bound on the degrees is not uniform and it depends on the directions. In this subsection, we will adopt a factorization trick due to Xie [Xie15] to overcome this difficulty, but in the loss of slightly worse bound. First, we began with the following lemma observed by Xie:

**Lemma 3.6.** *For all positive integers  $\tilde{d}_0$  every integer  $d \geq \tilde{d}_0^2 + \tilde{d}_0$  can be decomposed into*

$$d = (\tilde{d}_0 + 1)a + (\tilde{d}_0 + 2)b$$

where  $a$  and  $b$  are nonnegative integers.

Let  $X$  be an  $n$ -dimensional smooth projective variety, equipped with a very ample line bundle  $A$ . Let  $c$  be any integer satisfying  $1 \leq c \leq \lceil \frac{n}{2} \rceil$ . Set  $k_0 = \lceil \frac{n}{c} \rceil - 1$ ,  $\delta_0 := n(k_0 + 1)$ ,  $r_0 := c(k_0 + 1)\delta_0^{c(k_0+1)-1}(1 + k_0 + k_0\delta_0) + 1$ , and  $\tilde{d}_0 := \delta_0(r_0 + k_0) + k_0 - 1$ . Then any  $c$ -tuple of integers in the form  $(\tilde{d}_0 + 1, \dots, \tilde{d}_0 + 1, \tilde{d}_0 + 2, \dots, \tilde{d}_0 + 2)$  satisfies the condition (3.11). Take  $Z$  to be any complete intersection of  $c$  general hypersurfaces in  $|A^{\tilde{d}_0+1}|$  or  $|A^{\tilde{d}_0+2}|$ , and  $\hat{Z}_k$  is the variety obtained by the

blow-up of  $Z_k$  along the  $k$ -th asymptotic Wronskian ideal sheaf  $\mathfrak{w}_\infty(Z_k)$ . From Section 2.4 we see that, the Wronskian ideal sheaf is functorial under restrictions and thus  $\hat{Z}_k = \nu_k^{-1}(Z_k)$ , where  $\nu_k : \hat{X}_k \rightarrow X_k$  is also the blow-up of the Wronskian ideal sheaf  $\mathfrak{w}_\infty(X_k)$ . From Theorem 3.2 and Claim 3.3 we see that, the line bundle

$$\nu_k^*(\mathcal{O}_{X_k}(c\delta_0^{c(k_0+1)-1}k') \otimes \pi_{0,k}^*A^{-1}) \otimes \mathcal{O}_{\hat{X}_k}(-c\delta_0^{c(k_0+1)-1}F)|_{\hat{Z}_k}$$

is nef. Take an ample line bundle on  $\hat{X}_k$  of the form

$$\nu_k^*(\mathcal{O}_{X_k}(a_1, \dots, a_k) \otimes \pi_{0,k}^*A^{a_0}) \otimes \mathcal{O}_{\hat{X}_k}(-F)$$

where  $a_0, \dots, a_k \in \mathbb{N}$ . Then the line bundle

$$\nu_k^*(\mathcal{O}_{X_k}(a_1, \dots, a_{k-1}, a_k + a_0c\delta_0^{c(k_0+1)-1}k')) \otimes \mathcal{O}_{\hat{X}_k}(-(a_0c\delta_0^{c(k_0+1)-1} + 1)F)|_{\hat{Z}_k}$$

is ample. Within this setting, we have

**Theorem 3.3.** *For any  $c$ -tuple  $\mathbf{d} := (d_1, \dots, d_c)$  such that  $d_p \geq \tilde{d}_0^2 + \tilde{d}_0$  for each  $1 \leq p \leq c$ , for general hypersurfaces  $H_p \in |A^{d_p}|$ , their complete intersection  $Z := H_1 \cap \dots \cap H_c$  has almost  $k$ -jet ampleness provided that  $k \geq k_0$ .*

*Moreover, there exists a uniform  $(e_0, e_1, \dots, e_c) \in \mathbb{N}^{c+1}$  which does not depend on  $\mathbf{d}$ , such that*

$$\nu_k^*(\mathcal{O}_{Z_k}(e_1, \dots, e_c)) \otimes \mathcal{O}_{\hat{Z}_k}(-e_0F_{Z_k})$$

*is an ample line bundle, where where  $\nu_k : \hat{Z}_k \rightarrow Z_k$  is also the blow-up of the Wronskian ideal sheaf  $\mathfrak{w}_\infty(Z_k)$ , and  $F_{Z_k}$  is the effective cartier divisor on  $\hat{Z}_k$  such that  $\mathcal{O}_{\hat{Z}_k}(-F_{Z_k}) = \nu_k^*\mathfrak{w}_\infty(Z_k)$ .*

*Proof.* Let us denote by  $q : \mathcal{Z}_{\mathbf{d}} \rightarrow \prod_{p=1}^c |A^{d_p}|$  the universal family of  $c$ -complete intesections of hypersurfaces in  $\prod_{p=1}^c |A^{d_p}|$ , i.e.

$$\mathcal{Z}_{\mathbf{d}} := \{(s_1, \dots, s_c; x) \in \prod_{p=1}^c |A^{d_p}| \times X | \forall p, s_p \in |A^{d_p}| \text{ and } s_p(x) = 0\}. \quad (3.12)$$

By Lemma 3.6 we have the following decompositions

$$d_p = (\tilde{d}_0 + 1)a_p + (\tilde{d}_0 + 2)b_p$$

for each  $1 \leq p \leq c$ . Consider the linear system  $V_p \subset |A^{d_p}|$  generated by sections in  $\text{Sym}^{a_p}|A^{\tilde{d}_0+1}| \times \text{Sym}^{b_p}|A^{\tilde{d}_0+2}|$ , then for a generic choice of  $(s_1, \dots, s_c) \in V_1 \times \dots \times V_c$ , their complete intersection  $Y = \sum_{s=1}^l n_s Z^s$  (may not be reduced) is a union of smooth codimension  $c$  subvarieties  $Z^1, \dots, Z^l$  which are complete intersections of  $c$  general hypersurfaces in  $|A^{\tilde{d}_0+1}|$  or  $|A^{\tilde{d}_0+2}|$ . By the arguments above the line bundle

$$\nu_k^*(\mathcal{O}_{X_k}(a_1, \dots, a_{k-1}, a_k + a_0c\delta_0^{c(k_0+1)-1}k')) \otimes \mathcal{O}_{\hat{X}_k}(-(a_0c\delta_0^{c(k_0+1)-1} + 1)F)|_{\hat{Z}_k^s}$$

is ample for each  $s = 1, \dots, l$ , and so is for  $Y$ . Since ampleness is open in families, this also holds for the general fiber  $Z$  of  $q : \mathcal{Z}_{\mathbf{d}} \rightarrow \prod_{p=1}^c |A^{d_p}|$ , that is, for the complete intersection  $Z := H_1 \cap \dots \cap H_c$  of any general hypersurfaces in  $\prod_{p=1}^c |A^{d_p}|$ , the line bundle

$$\begin{aligned} & \mathcal{O}_{Z_k}(a_1, \dots, a_{k-1}, a_k + a_0c\delta_0^{c(k_0+1)-1}k') \otimes \mathcal{O}_{\hat{Z}_k}(-(a_0c\delta_0^{c(k_0+1)-1} + 1)F_{Z_k}) = \\ & \nu_k^*(\mathcal{O}_{X_k}(a_1, \dots, a_{k-1}, a_k + a_0c\delta_0^{c(k_0+1)-1}k')) \otimes \mathcal{O}_{\hat{X}_k}(-(a_0c\delta_0^{c(k_0+1)-1} + 1)F)|_{\hat{Z}_k} \end{aligned}$$

is ample. As the choice of  $\mathbf{a}$  is independant of  $\mathbf{d}$ , we obtain our theorem.  $\square$

Roughly, we can take the lower bound to be  $c^2 n^{2n+2c}(\lceil \frac{n}{c} \rceil)^{2n+2c+4} \geq \tilde{d}_0^2 + \tilde{d}_0$ , and we finish the proof of Theorem B.

### 3.5 ON THE DIVERIO-TRAPANI CONJECTURE

In this subsection we will prove Theorem C. Let  $X$  be a projective manifold of dimension  $n$  and  $A$  a very ample line bundle on  $X$ . Recall that we denote by  $\hat{X}_k$  the blow-up of  $X_k$  along the asymptotic Wronskian ideal sheaf  $\mathfrak{w}_\infty(X_k)$ , and  $F$  the effective Cartier divisor such that  $\mathcal{O}_{\hat{X}_k}(-F) = \nu_k^*(\mathfrak{w}_\infty(X_k))$ . From the proof of Theorem 3.3, one can find a uniform  $\mathbf{e} := (e_0, \dots, e_c) \in \mathbb{N}^{c+1}$  such that, for the generic fiber  $Z$  of the universal family  $q : \mathcal{Z}_d \rightarrow \prod_{p=1}^c |A^{d_p}|$  defined in (3.12), where  $d_p \geq \tilde{d}_0^2 + \tilde{d}_0$  for each  $1 \leq p \leq c$ , the line bundle

$$\nu_k^* \mathcal{O}_{X_k}(e_1, \dots, e_c) \otimes \mathcal{O}_{\hat{X}_k}(-e_0 F)|_{\hat{Z}_k} = \nu_k^* \mathcal{O}_{Z_k}(e_1, \dots, e_c) \otimes \mathcal{O}_{\hat{Z}_k}(-e_0 F_{Z_k}) \quad (3.13)$$

is *very* ample. From Section 2.3 we can take an open covering  $\{U_\alpha\}$  of  $Z$  such that:

- a) each  $U_{\alpha k} := \pi_{0,k}^{-1}(U_\alpha)$  is a trivial product  $U_\alpha \times \mathcal{R}_{n-c,k}$ , where  $\mathcal{R}_{n-c,k}$  is some smooth rational variety.
- b) Set  $\text{pr}_2 : U_\alpha \times \mathcal{R}_{n-c,k} \rightarrow \mathcal{R}_{n-c,k}$  to be the projection map. There exists an ideal sheaf  $\mathcal{I}_{n-c,k}$  on  $\mathcal{R}_{n-c,k}$  such that

$$\mathfrak{w}_\infty(Z_k) = \text{pr}_2^*(\mathcal{I}_{n-c,k}).$$

Let us denote by  $\mu_k : \hat{\mathcal{R}}_{n-c,k} \rightarrow \mathcal{R}_{n-c,k}$  the blow-up of  $\mathcal{R}_{n-c,k}$  along  $\mathcal{I}_{n-c,k}$ , and  $E$  is the effective divisor on  $\hat{\mathcal{R}}_{n-c,k}$  such that

$$\mathcal{O}_{\hat{\mathcal{R}}_{n-c,k}}(-E) := \mu_k^*(\mathcal{I}_{n-c,k}).$$

Set  $\hat{U}_{\alpha k} := \nu_k^{-1}(U_{\alpha k})$ , then we have

$$\begin{array}{ccc} U_\alpha \times \hat{\mathcal{R}}_{n-c,k} & \xrightarrow{\cong} & \hat{U}_{\alpha k} \\ \downarrow 1 \times \mu_k & & \downarrow \nu_k \\ U_\alpha \times \mathcal{R}_{n-c,k} & \xrightarrow{\cong} & U_{\alpha k}. \end{array}$$

Therefore,  $\pi_{0,k} \circ \nu_k : \hat{Z}_k \rightarrow Z$  is a local isotrivial family with fiber  $\hat{\mathcal{R}}_{n-c,k}$ , and thus for any  $j > 0$  the direct image  $(\pi_{0,k} \circ \nu_k)_*(jL)$  is always locally free on  $Z$ , here we denote by  $L := \nu_k^* \mathcal{O}_{Z_k}(e_1, \dots, e_c) \otimes \mathcal{O}_{\hat{Z}_k}(-e_0 F_{Z_k})$ . Since

$$(\nu_k)_*(jL) = \mathcal{O}_{Z_k}(j e_1, \dots, j e_c) \otimes \mathcal{I}_j,$$

where  $\mathcal{I}_j = (\nu_k)_* \mathcal{O}_{\hat{Z}_k}(-j e_0 F_{Z_k})$  is some ideal sheaf of  $Z_k$  supported on  $Z_k^{\text{Sing}}$ , by the Direct image formula (2.3) we have

$$(\pi_{0,k} \circ \nu_k)_*(jL) \subset \overline{F}^{j\mathbf{e}} E_{k,jm} T_Z^* \quad (3.14)$$

where  $m = e_1 + \dots + e_c$ .

**Claim 3.4.** *There exists a positive integer  $j_1$  such that for each  $j \geq j_1$ , the direct image  $(\pi_{0,k} \circ \nu_k)_*(jL) \subset \mathcal{O}(\overline{F}^{j\mathbf{e}} E_{k,jm} T_Z^*)$  is an ample vector bundle on  $Z$ .*

**Proof:** Let us denote by  $A_Z := A|_Z$ . As  $L$  is ample, one can find an integer  $j_0 \gg 0$  such that for each  $j \geq j_0$ , all higher direct image sheaf  $R^i(\pi_{0,k} \circ \nu_k)_*(jL)$  vanishes, and  $jL - (\pi_{0,k} \circ \nu_k)^* A_Z$  is ample.

Set  $V_j := (\pi_{0,k} \circ \nu_k)_*(jL - (\pi_{0,k} \circ \nu_k)^* A_Z)$  which is a local free sheaf for any  $j \geq 0$ . Consider any coherent  $\mathcal{F}$  on  $Z$ . Then by the degeneration of the Leray spectral sequence, for each  $j \geq j_0$ , we have

$$H^i(Z, V_j \otimes \mathcal{F}) = H^i(\hat{Z}_k, L^j \otimes (\pi_{0,k} \circ \nu_k)^* A_Z^{-1} \otimes (\pi_{0,k} \circ \nu_k)^* \mathcal{F}) \quad (3.15)$$

for any  $i > 0$ . Fix a point  $y \in Z$ , with maximal ideal  $\mathcal{M}_y \subset \mathcal{O}_Z$ . Then we have the exact sequence

$$0 \rightarrow \mathcal{M}_y \rightarrow \mathcal{O}_Z \rightarrow \mathcal{O}_Z / \mathcal{M}_y \rightarrow 0.$$

As  $L$  is ample, there exists a positive integer  $j(y) \geq j_0$  such that

$$H^1(Z, V_j \otimes \mathcal{M}_y) = H^1(\hat{Z}_k, L^j \otimes (\pi_{0,k} \circ \nu_k)^* A_Z^{-1} \otimes (\pi_{0,k} \circ \nu_k)^* \mathcal{M}_y) = 0$$

for  $j \geq j(y)$ , and so we see using the exact sequence above that  $V_j$  is generated by its global sections at  $y$ . The same therefore holds in a Zariski open neighborhood of  $y$ , and by the compactness of  $Z$  we can find a integer  $j_1 \geq j_0$  such that  $V_j$  is globally generated when  $j \geq j_1$ . Thus  $V_j \otimes A_Z = (\pi_{0,k} \circ \nu_k)_*(jL)$  is an ample vector bundle for any  $j \geq j_1$ .  $\blacksquare$

Since the ampleness is open in families, then Claim 3.4 holds for general fibers of  $q : \mathcal{Z}_{\mathbf{d}} \rightarrow \prod_{p=1}^c |A^{d_p}|$ . Set  $U \subset \prod_{p=1}^c |A^{d_p}|$  to be a Zariski open set of  $\prod_{p=1}^c |A^{d_p}|$  such that when restricted to  $\mathcal{Y} := q^{-1}(U)$ ,  $q$  is a smooth fibration. Denote by  $\mathcal{Y}_k$  the  $k$ -th Demailly-Semple tower of  $(\mathcal{Y}, T_{\mathcal{Y}/U})$ , and  $\nu_k : \hat{\mathcal{Y}}_k \rightarrow \mathcal{Y}_k$  the blowing-up of the asymptotic Wronskian ideal sheaf  $\mathfrak{w}_{\infty}(\mathcal{Y}_k)$  with  $\mathcal{O}_{\hat{\mathcal{Y}}_k}(-F_{\mathcal{Y}_k}) = \nu_k^* \mathfrak{w}_{\infty}(\mathcal{Y}_k)$ . Then for every  $j \gg 0$ , we define the vector bundle  $V_j$  on  $\mathcal{Y}$  by

$$V_j := (\pi_{0,k} \circ \nu_k)_*(\nu_k^* \mathcal{O}_{\mathcal{Y}_k}(je_1, \dots, je_n) \otimes \mathcal{O}_{\hat{\mathcal{Y}}_k}(-je_0 F_{\mathcal{Y}_k})),$$

and its restriction to the general fiber  $Z$  of  $q$  is

$$(\pi_{0,k} \circ \nu_k)_*(\nu_k^* \mathcal{O}_{Z_k}(je_1, \dots, je_n) \otimes \mathcal{O}_{\hat{Z}_k}(-je_0 F_{Z_k})),$$

which is ample by Claim 3.4. We finish the proof of the first part in Theorem C. Since  $L = \nu_k^* \mathcal{O}_{Z_k}(e_1, \dots, e_c) \otimes \mathcal{O}_{\hat{Z}_k}(-e_0 F_{Z_k})$  is very ample on  $\hat{Z}_k$ , we can take  $j \gg 0$  such that  $jL - (\pi_{0,k} \circ \nu_k)^* A_Z^{-1}$  is still very ample, and by the relation

$$(\nu_k)_*(jL) = \mathcal{O}_{Z_k}(je_1, \dots, je_k) \otimes \mathcal{I}_j,$$

we see that the base locus of

$$H^0(Z_k, \mathcal{O}_{Z_k}(je_1, \dots, je_k) \otimes (\pi_{0,k})^* A_Z^{-1} \otimes \mathcal{I}_j)$$

is contained in  $Z_k^{\text{Sing}}$ . We finish the proof of Theorem C.

## 4 EFFECTIVE ESTIMATES RELATED TO THE NAKAMAYE THEOREM

In this section we prove Theorem 3.1. For simplicity and to make this part readable, we give a complete proof for  $c = 1$ . The proof for the general cases is exact the same and we will show the general ideas for that. We begin with some definitions and notations of the *universal Grassmannian*.

We consider  $V := H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(\delta))$ , that is, the space of homogeneous polynomials of degree  $\delta$  in  $\mathbb{C}[z_0, \dots, z_N]$ , and for any  $J \subset \{0, \dots, N\}$ , we set

$$\mathbb{P}_J := \{[z_0, \dots, z_N] \in \mathbb{P}^N \mid z_j = 0 \text{ if } j \in J\}.$$

Given any  $\Delta \in \text{Gr}_{k+1}(V)$  and  $[z] \in \mathbb{P}^N$ , we write  $\Delta([z]) = 0$  if and only if  $P(z) = 0$  for all  $P \in \Delta \subset V$ . We then define the universal Grassmannian to be

$$\mathcal{Y} := \{(\Delta, [z]) \in \text{Gr}_{k+1}(V) \times \mathbb{P}^N \mid \Delta([z]) = 0\}, \quad (4.1)$$

and for any  $J \subset \{0, \dots, N\}$ , set

$$\mathcal{Y}_J := \mathcal{Y} \cap (\text{Gr}_{k+1}(V) \times \mathbb{P}_J). \quad (4.2)$$

From now on we always assume that  $k+1 \geq N$ , then  $p : \mathcal{Y} \rightarrow \text{Gr}_{k+1}(V)$  is a generically finite to one morphism. Denote  $q : \mathcal{Y} \rightarrow \mathbb{P}^N$  to be the projection on the second factor. Let  $\mathcal{L}$  be the very ample line bundle on  $\text{Gr}_{k+1}(V)$  which is the pull back of  $\mathcal{O}(1)$  under the Plücker embedding. Then  $p^* \mathcal{L}|_{\mathcal{Y}_J}$  is a big and nef line bundle on  $\mathcal{Y}_J$  for any  $J$ . For any  $J \subset \{0, \dots, N\}$  we denote by  $p_J : \mathcal{Y}_J \rightarrow \text{Gr}_{k+1}(V)$ , and  $q_J : \mathcal{Y}_J \rightarrow \mathbb{P}_J$  the projections. Similarly we set

$$E_J := \{y \in \mathcal{Y} \mid \dim_y(p_J^{-1}(p_J(y))) > 0\}$$

$$G_J^{\infty} := p_J(E_J) \subset \text{Gr}_{k+1}(V),$$

then  $E_J = \text{Null}(p^*\mathcal{L}|_{\mathcal{Y}_J})$ . For  $J = \emptyset$  we have  $\mathcal{Y}_J = \mathcal{Y}$  and denote by  $E := E_\emptyset$  and  $G^\infty := G_\emptyset^\infty$ .

Now we begin to prove Theorem 3.1. First of all suppose that  $c = 1$  and  $k + 1 = N$ . Then in this case  $p : \mathcal{Y} \rightarrow \text{Gr}_N(V)$  is a generically finite to one surjective morphism. We first deal with the case  $J = \emptyset$ .

Let us pick a smooth curve  $C$  in  $\text{Gr}_N(V)$  of degree 1, given by

$$\Delta([t_0, t_1]) := \text{Span}(z_1^\delta, z_2^\delta, \dots, z_{N-1}^\delta, t_0 z_N^\delta + t_1 z_0^\delta),$$

where  $[t_0, t_1] \in \mathbb{P}^1$ . Indeed, the curve  $C$  is the line in the projective space  $\mathbb{P}(\Lambda^N V)$  defined by two vectors  $z_0^\delta \wedge z_1^\delta \wedge \dots \wedge z_{N-1}^\delta$  and  $z_1^\delta \wedge z_2^\delta \wedge \dots \wedge z_N^\delta$  in  $\Lambda^N V$ , which is of degree 1 with respect to the tautological line bundle  $\mathcal{L}$ . That is,

$$\mathcal{L} \cdot C = 1.$$

Now consider the hyperplane  $D$  in  $\mathbb{P}^N$  given by  $\{[z_0, \dots, z_N] | z_0 + z_N = 0\}$ . We have

**Lemma 4.1.** *The intersection number of the curve  $p^*C$  and the divisor  $q^*D$  in  $\mathcal{Y}$  is  $\delta^{N-1}$ . Moreover,  $p_*q^*D \sim \delta^{N-1}\mathcal{L}$ , where “ $\sim$ ” stands for linear equivalence.*

*Proof.* An easy computation shows that  $p^*C$  and  $q^*D$  intersect only at the point

$$\text{Span}(z_1^\delta, z_2^\delta, \dots, z_{N-1}^\delta, z_N^\delta + (-1)^{\delta+1}z_0^\delta) \times [1, 0, \dots, 0, -1] \in \mathcal{Y}$$

with multiplicity  $\delta^{N-1}$ . The first statement follows. By the projection formula we have

$$p_*q^*D \cdot C = p_*(q^*D \cdot p^*C) = \delta^{N-1}.$$

As  $\text{Pic}(\text{Gr}_N(V)) \cong \mathbb{Z}$  with the generator  $\mathcal{L}$ , then we get  $p_*q^*D \sim \delta^{N-1}\mathcal{L}$  by the fact that  $\mathcal{L} \cdot C = 1$ .  $\square$

We first observe that  $p^*p_*q^*D - q^*D$  is an effective divisor of  $\mathcal{Y}$ , and by Lemma 4.1 we conclude that  $\delta^{N-1}p^*\mathcal{L} - q^*\mathcal{O}_{\mathbb{P}^N}(1)$  is effective. We also have a good control of the base locus as follows:

**Claim 4.1.** *For any  $m \geq \delta^N$ , we always have*

$$\text{Bs}(mp^*\mathcal{L} - q^*\mathcal{O}_{\mathbb{P}^N}(1)) \subset p^{-1}(G^\infty). \quad (4.3)$$

Proof: Pick any  $\Delta_0 \notin G^\infty$ ,  $p^{-1}(\Delta_0)$  is a finite set by the definition of  $G^\infty$ . Thus one can choose a hyperplane  $D \in H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1))$  such that  $D \cap q(p^{-1}(\Delta_0)) = \emptyset$ . From Lemma 4.1 we know that the divisor  $p^*p_*q^*D - q^*D$  is effective and lies in the linear system  $|\delta^{N-1}p^*\mathcal{L} - q^*\mathcal{O}_{\mathbb{P}^N}(1)|$  of  $\mathcal{Y}$ .

For any  $\Delta \in \text{Gr}_N(V)$ , if we denote by

$$\text{Int}(\Delta) := \{[z] \in \mathbb{P}^N | \Delta([z]) = 0\},$$

then  $q(p^{-1}(\Delta)) = \text{Int}(\Delta)$ . Hence the condition that  $D \cap q(p^{-1}(\Delta_0)) = \emptyset$  is equivalent to that  $\text{Int}(\Delta_0) \cap D = \emptyset$ . However, for any  $\Delta \in p_*q^*D$ , we must have  $\text{Int}(\Delta) \cap D \neq \emptyset$ , therefore  $\Delta_0 \notin p_*q^*D$ . As  $\Delta_0$  was arbitrary, we conclude that

$$\text{Bs}(\delta^{N-1}p^*\mathcal{L} - q^*\mathcal{O}_{\mathbb{P}^N}(1)) \subset p^{-1}(G^\infty).$$

As  $\mathcal{L}$  is very ample on  $\text{Gr}_N(V)$ , for any  $m \geq \delta^{N-1}$ , we have

$$\text{Bs}(mp^*\mathcal{L} - q^*\mathcal{O}_{\mathbb{P}^N}(1)) \subset \text{Bs}(\delta^{N-1}p^*\mathcal{L} - q^*\mathcal{O}_{\mathbb{P}^N}(1)) \subset p^{-1}(G^\infty).$$

The Claim is thus proved.  $\blacksquare$

Now we deal with the general case  $J \subset \{0, \dots, N\}$ . Without loss of generality we can assume that  $J = \{n+1, \dots, N\}$ . First recall our previous notation  $p_J : \mathcal{Y}_J \rightarrow \text{Gr}_N(V)$ , and let  $q_J : \mathcal{Y}_J \rightarrow \mathbb{P}_J$  be the second projection. For any  $\Delta_0 \notin G_J^\infty$ , the set  $p_J^{-1}(\Delta_0) = \text{Int}(\Delta_0) \cap \mathbb{P}_J$  is finite. Thus one can choose a generic hyperplane  $D \in H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1))$  such that  $\text{Int}(\Delta_0) \cap D \cap \mathbb{P}_J = \emptyset$ . One can further choose a proper coordinate for  $\mathbb{P}^N$  such that  $D = \{z_n = 0\}$ .



Observe that  $\mathcal{Y} \xrightarrow{q} \mathbb{P}^N$  is a *local trivial fibration*. Indeed, any linear transformation  $g \in \mathrm{GL}(\mathbb{C}^{N+1})$  induces a natural action  $\tilde{g} \in \mathrm{GL}(V)$ , hence also a biholomorphism  $\hat{g}$  of  $\mathrm{Gr}_N(V)$ . For any  $e \in \mathbb{P}^N$ ,  $\hat{g}$  maps the fiber  $q^{-1}(e)$  to  $q^{-1}(g(e))$  bijectively. Since  $\mathrm{GL}_{N+1}(\mathbb{C})$  acts transitively on  $\mathbb{P}^N$ , the fibration  $\mathcal{Y} \xrightarrow{q} \mathbb{P}^N$  can then be locally trivialized. Therefore  $q_J^*(D \cap \mathbb{P}_J)$  is a *reduced* divisor in  $\mathcal{Y}_J$ . Set  $E := p_J(q_J^{-1}(D \cap \mathbb{P}_J))$  set-theoretically. Then for any divisor  $\tilde{H} \in |m\mathcal{L}|$  on  $\mathrm{Gr}_N(V)$  such that  $E \subset \tilde{H}$  and  $p_J(\mathcal{Y}_J) \not\subset \tilde{H}$ ,  $p_J^*(\tilde{H}) - q_J^*(D \cap \mathbb{P}_J)$  is an effective divisor in  $|mp_J^*\mathcal{L} - q_J^*\mathcal{O}_{\mathbb{P}_J}(1)|$ . However, it may happen that for any hyperplane  $\tilde{D} \in \mathbb{P}^N$ , all constructed divisors of the form  $p_*q^*(\tilde{D})$  will always contain  $\Delta_0$ .

Choose a decomposition of  $V = V_1 \oplus V_2$  such that  $V_1$  is spanned by the vectors  $\{z^\alpha \in V | \alpha_n = \dots = \alpha_N = 0\}$  and  $V_2$  is spanned by other  $z^\alpha$ 's. Let us denote  $G$  to be the subgroup of the general linear group  $GL(V)$  which is the lower triangle matrix with respect to the decomposition of  $V = V_1 \oplus V_2$  as follows:

$$\{g \in GL(V) | g = \begin{bmatrix} I & 0 \\ A & B \end{bmatrix}, B \in GL(V_2), A \in \mathrm{Hom}(V_1, V_2)\}. \quad (4.4)$$

The subgroup  $G$  also induced a natural group action on the Grassmannian  $\mathrm{Gr}_N(V)$ , and we have the following

**Lemma 4.2.** *Set  $H := p_*(q^*D)$ . Then for any  $g \in G$ ,  $E \subset g(H)$  and there exists a  $g_0 \in G$  such that  $\Delta_0 \notin g_0(H)$ .*

*Proof.* For any  $\Delta \in \mathrm{Gr}_N(V)$ , choose  $\{s_1, \dots, s_N\} \subset V$  which spans  $\Delta$ . Let  $s_i = u_i + v_i$  be the unique decomposition of  $s_i$  under  $V = V_1 \oplus V_2$ . Then by  $E := p_J(q_J^{-1}(D \cap \mathbb{P}_J))$  we see that  $\Delta \in E$  if and only if  $\cap_{i=1}^N \{u_i = 0\} \cap \mathbb{P}^{n-1} \neq \emptyset$ , where  $\mathbb{P}^{n-1} := \{[z_0 : \dots : z_N] \in \mathbb{P}^N | z_j = 0 \text{ for } j \geq n\}$ . For any  $g \in GL(V)$ ,  $g(\Delta)$  is spanned by  $\{g(s_1), \dots, g(s_N)\}$ . By the definition of  $G$ , for any  $g \in G$ , we have the decomposition  $g(s_i) = u_i + v'_i$  with respect to  $V = V_1 \oplus V_2$  which keeps the  $V_1$  factors invariant. Thus we prove the first part of the claim.

Set  $\{t_1, \dots, t_N\} \subset V$  which spans  $\Delta_0$  and  $t_i = u_i + v_i$  be the unique decomposition of  $t_i$  under  $V = V_1 \oplus V_2$ . Since  $\mathrm{Int}(\Delta_0) \cap \mathbb{P}^{n-1} = \emptyset$ , we have  $\cap_{i=1}^N \{u_i = 0\} \cap \mathbb{P}^{n-1} = \emptyset$ . We can then choose the basis  $\{t_1, \dots, t_N\}$  spanning  $\Delta_0$  properly, so that

- (i)  $\cap_{i=1}^n \{u_i = 0\} \cap \mathbb{P}^{n-1} = \emptyset$ ;
- (ii) for some  $m \geq n$ ,  $\{u_1, \dots, u_m\}$  is a set of vectors in  $V_1$  which is linearly independant;
- (iii)  $u_{m+1} = \dots = u_N = 0$ .

Then  $\cap_{i=1}^n \{u_i = 0\} \cap \{z_n = 0\} = \mathbb{P}^{N-n-1} := \{[z_0 : \dots : z_N] \in \mathbb{P}^N | z_j = 0 \text{ for } j \leq n\}$ , and  $\{v_{m+1}, \dots, v_N\}$  is a set of linearly independant vectors in  $V_2$ .

Let us denote by  $\Delta' \in \mathrm{Gr}_N(V)$  spanned by

$$\begin{cases} \tilde{u}_1 := u_1 \\ \vdots \\ \tilde{u}_n := u_n \\ \tilde{u}_{n+1} := u_{n+1} + z_{n+1}^\delta \\ \vdots \\ \tilde{u}_m := u_m + z_m^\delta \\ \tilde{u}_{m+1} := u_{m+1} + z_{m+1}^\delta = z_{m+1}^\delta \\ \vdots \\ \tilde{u}_N := u_N + z_N^\delta = z_N^\delta \end{cases}.$$

Then  $\mathrm{Int}(\Delta') \cap \{z_n = 0\} = \emptyset$ , which is equivalent to that  $\Delta' \notin H := p_*q^*(D)$ . By the choice of  $\Delta'$  one can find a  $g_0 \in G$  such that  $g_0(\Delta') = \Delta_0$ . Indeed, by linear independances of  $\{v_{m+1}, \dots, v_N\}$  in

$V_2$  and  $\{u_1, \dots, u_m\}$  in  $V_1$ , we can find a  $B \in GL(V_2)$  satisfying that  $B(z_i^\delta) = v_i$  for all  $i \geq m+1$ , and  $A \in \text{Hom}(V_1, V_2)$  such that  $A(u_i) = v_i$  for  $i \leq n$  and  $A(u_j) = v_j - B(z_j^\delta)$  for  $n+1 \leq j \leq m$ . Set  $g_0 := \begin{bmatrix} I & 0 \\ A & B \end{bmatrix}$  which is the type (4.4), and by the construction of  $g_0$  we have that  $g_0(\Delta') = \Delta_0$ . Thus  $\Delta_0 \notin g_0(H)$  and we finish the proof of the claim.  $\square$

Since  $H \in |\delta^{N-1}\mathcal{L}|$ ,  $g_0(H)$  still lies in  $|\delta^{N-1}\mathcal{L}|$ . Indeed, since the complex general linear group  $GL(V)$  is connected, the automorphism map of  $\text{Gr}_N(V)$  induced by  $g_0$ -action is homotopic to the identity map, and thus the  $g_0$  action induces the identity on the cohomology groups. By Lemma 4.2,  $E \subset g_0(H)$  and  $\Delta_0 \notin g_0(H)$ . As  $q_J^*(D \cap \mathbb{P}_J)$  is a reduced (Cartier) divisor on  $\mathcal{Y}_J$ , the divisor

$$p_J^*(g_0(H)) - q_J^*(D \cap \mathbb{P}_J) \in |\delta^{N-1}p_J^*\mathcal{L} - q_J^*\mathcal{O}_{\mathbb{P}_J}(1)|$$

is effective and avoids the finite set  $p_J^{-1}(\Delta_0)$ .

Since  $\Delta_0 \in \text{Gr}_N(V)$  is any arbitrary point not contained in  $G_J^\infty$ , thus the base locus of  $\delta^{N-1}p_J^*\mathcal{L} - q_J^*\mathcal{O}_{\mathbb{P}_J}(1)$  is totally contained in  $p_J^{-1}(G_J^\infty)$ . In conclusion we have the following theorem:

**Theorem 4.1.** *Let  $\mathcal{Y} \subset \text{Gr}_N(V) \times \mathbb{P}^N$  and  $\mathcal{Y}_J$  be the universal families defined in (4.1) and (4.2). For any  $J \subset \{0, \dots, N\}$ , we have*

$$\text{Bs}(mp^*\mathcal{L} - q^*\mathcal{O}_{\mathbb{P}^N}(1)|_{\mathcal{Y}_J}) \subset p_J^{-1}(G_J^\infty)$$

for any  $m \geq \delta^{N-1}$ .

Fix any positive integer  $n < N$ . Consider  $\mathbb{P}^n$  as a subspace of  $\mathbb{P}^N$  defined by  $z_{n+1} = \dots = z_N = 0$ . Set  $V_n := H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\delta))$ , and we have a natural inclusion  $\text{Gr}_N(V_n) \subset \text{Gr}_N(V)$ . For any  $J \subset \{0, \dots, n\}$ , we denote by  $\tilde{J} := J \cup \{n+1, \dots, N\}$ , and  $\mathbb{P}_{\tilde{J}} := \{[z_0, \dots, z_N] \in \mathbb{P}^N | z_j = 0 \text{ if } j \in \tilde{J}\}$ . Set

$$\tilde{\mathcal{Y}}_J := \{(\Delta, [z]) \in \text{Gr}_N(V_n) \times \mathbb{P}_{\tilde{J}} | \Delta([z]) = 0\}.$$

Define  $\tilde{p}_J : \tilde{\mathcal{Y}}_J \rightarrow \text{Gr}_N(V_n)$  and  $\tilde{q}_J : \tilde{\mathcal{Y}}_J \rightarrow \mathbb{P}_{\tilde{J}}$  the respective projections. Set

$$\tilde{G}_J^\infty := \{\Delta \in \text{Gr}_N(V_n) | \tilde{p}_J^{-1}(\Delta) \text{ is not finite set}\}.$$

Let  $\mathcal{Y} \subset \text{Gr}_N(V) \times \mathbb{P}^N$  and  $\mathcal{Y}_J$  be the universal families defined in (4.1) and (4.2). There is a natural inclusion  $i_n : \text{Gr}_N(V_n) \hookrightarrow \text{Gr}_N(V)$ , which induces the following inclusions:

$$\begin{array}{ccc} \tilde{\mathcal{Y}}_J & \hookrightarrow & \text{Gr}_N(V_n) \times \mathbb{P}_{\tilde{J}} \\ \downarrow & & \downarrow i_n \times \mathbb{1} \\ \mathcal{Y}_J & \hookrightarrow & \text{Gr}_N(V) \times \mathbb{P}_{\tilde{J}} \end{array}$$

Under the inclusion  $i_n$ , we have

$$\tilde{G}_J^\infty = G_J^\infty \cap \text{Gr}_N(V_n).$$

From Theorem 4.1, for  $m \geq \delta^{N-1}$  we also have

$$\begin{aligned} \text{Bs}(mp^*\mathcal{L} - q^*\mathcal{O}_{\mathbb{P}^N}(1)|_{\tilde{\mathcal{Y}}_J}) &\subset \text{Bs}(mp^*\mathcal{L} - q^*\mathcal{O}_{\mathbb{P}^N}(1)|_{\mathcal{Y}_J}) \cap \tilde{\mathcal{Y}}_J \\ &\subset p_J^{-1}(G_J^\infty) \cap \tilde{\mathcal{Y}}_J \\ &= \tilde{p}_J^{-1}(\tilde{G}_J^\infty), \end{aligned} \tag{4.5}$$

where  $p_J : \mathcal{Y}_J \rightarrow \text{Gr}_N(V)$  and  $q_J : \mathcal{Y}_J \rightarrow \mathbb{P}_J$  are the projection maps. Since the pull back

$$i_n^* : \text{Pic}(\text{Gr}_N(V)) \xrightarrow{\sim} \text{Pic}(\text{Gr}_N(V_n))$$

is an isomorphism between the Picard groups, and  $\mathcal{L}_n := i_n^*\mathcal{L}$  is still the tautological line bundle on  $\text{Gr}_N(V_n)$ . Then

$$mp^*\mathcal{L} - q^*\mathcal{O}_{\mathbb{P}^N}(1)|_{\tilde{\mathcal{Y}}_J} = m\tilde{p}_J^*(\mathcal{L}_n) - \tilde{q}_J^*\mathcal{O}_{\mathbb{P}_J}(1),$$

and by (4.5) we have

$$\text{Bs}(m\tilde{p}_J^*(\mathcal{L}_n) - \tilde{q}_J^*\mathcal{O}_{\mathbb{P}_J}(1)) \subset \tilde{p}_J^{-1}(\tilde{G}_J^\infty). \quad (4.6)$$

We are in the situation to prove Theorem 3.1 for  $c = 1$  and general  $k + 1 \geq N$ :

**Theorem 4.2.** *For any  $k + 1 \geq N$ , set  $\mathcal{Y} \subset \text{Gr}_{k+1}(V) \times \mathbb{P}^N$  and  $\mathcal{Y}_J$  to be the universal families defined in (4.1) and (4.2). For any  $J \subset \{0, \dots, N\}$ , and  $k + 1 \geq N$ , we have*

$$\text{Bs}(mp^*\mathcal{L} - q^*\mathcal{O}_{\mathbb{P}^N}(1)|_{\mathcal{Y}_J}) \subset p_J^{-1}(G_J^\infty) \quad (4.7)$$

for any  $m \geq \delta^k$ .

*Proof.* Indeed, if we consider  $\mathbb{P}^N$  as a subspace in  $\mathbb{P}^{k+1}$  defined by  $z_{N+1} = \dots = z_{k+1} = 0$ , the theorem follows from (4.6) directly.  $\square$

The above theorem can be generalized to the case of products of Grassmannians. We first set  $N := c(k + 1)$ , and denote  $V_i := H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(\delta_i))$  and  $\mathbf{G} := \prod_{i=1}^c \text{Gr}_{k+1}(V_i)$  for simplicity. Let  $\mathcal{Y}$  be the generalized universal Grassmannian defined by

$$\mathcal{Y} := \{(\Delta_1, \dots, \Delta_c, z) \in \mathbf{G} \times \mathbb{P}^N \mid \forall i, \Delta_i([z]) = 0\}.$$

Let  $p : \mathcal{Y} \rightarrow \mathbf{G}$ ,  $q : \mathcal{Y} \rightarrow \mathbb{P}^N$  and  $p_i : \mathcal{Y} \rightarrow \text{Gr}_{k+1}(\delta_i)$  be the canonical projections to each factor; then  $p$  is a generically finite to one morphism. Define a group homeomorphism

$$\begin{aligned} \mathcal{L} : \mathbb{Z}^c &\rightarrow \text{Pic}(\mathbf{G}) \\ \mathbf{a} = (a_1, \dots, a_c) &\mapsto \mathcal{O}_{\text{Gr}_{k+1}(V_1)}(a_1) \boxtimes \dots \boxtimes \mathcal{O}_{\text{Gr}_{k+1}(V_c)}(a_c) \end{aligned}$$

which is moreover an isomorphism.

We then define smooth lines  $\{C_i\}_{i=1, \dots, c}$  in  $\mathbf{G}$ , given by

$$\begin{aligned} \Delta_i([t_0, t_1]) &:= \text{Span}(z_1^{\delta_1}, z_{c+1}^{\delta_1}, \dots, z_{kc+1}^{\delta_1}) \times \text{Span}(z_2^{\delta_2}, z_{c+2}^{\delta_2}, \dots, z_{kc+2}^{\delta_2}) \times \dots \\ &\times \text{Span}(t_0 z_i^{\delta_i} + t_1 z_0^{\delta_i}, z_{c+i}^{\delta_i}, \dots, z_{kc+i}^{\delta_i}) \times \dots \times \text{Span}(z_c^{\delta_c}, z_{2c}^{\delta_c}, \dots, z_{(k+1)c}^{\delta_c}) \end{aligned}$$

for  $[t_0, t_1] \in \mathbb{P}^1$ . It is easy to verify that  $\mathcal{L}(\mathbf{a}) \cdot C_i = a_i$  for each  $i$ . Consider the hyperplane  $D_i$  of  $\mathbb{P}^n$  given by  $\{[z_0, \dots, z_N] \mid z_i + z_0 = 0\}$ . Then we have

**Lemma 4.3.** *The intersection number of the curve  $p^*C_i$  and the divisor  $q^*D_i$  in  $\mathcal{Y}$  is  $b_i := \frac{\prod_{j=1}^c \delta_j^{k+1}}{\delta_i}$ . Moreover,  $p_*q^*\mathcal{O}_{\mathbb{P}^N}(1) \equiv \mathcal{L}(\mathbf{b})$ , where  $\mathbf{b} = (b_1, \dots, b_c)$ .*

*Proof.* It is easy to show that  $p^*C_i$  and  $q^*D_i$  intersect only at one point with multiplicity  $b_i$ . By the projection formula we have

$$p_*q^*D_i \cdot C_i = p_*(q^*D_i \cdot p^*C_i) = b_i.$$

Since

$$\mathcal{L}(\mathbf{a}) \cdot C_i = a_i$$

for any  $\mathbf{a} \in \mathbb{Z}^c$ . Thus

$$p_*q^*D_i \equiv p_*q^*\mathcal{O}_{\mathbb{P}^N}(1) \equiv \mathcal{L}(\mathbf{b}).$$

$\square$

Then by similar arguments above,  $p^*\mathcal{L}(\mathbf{b}) \otimes q^*\mathcal{O}_{\mathbb{P}^N}(-1)$  is effective, and its base locus

$$\text{Bs}(p^*\mathcal{L}(\mathbf{b}) \otimes q^*\mathcal{O}_{\mathbb{P}^N}(-1)) \subset p^{-1}(G^\infty), \quad (4.8)$$

where  $G^\infty$  is the set of points in  $\mathbf{G}$  such that their  $p$ -fiber is not a finite set. We can then apply the methods already used above to show that (4.8) also holds for all the strata  $\mathcal{Y}_I$  of  $\mathcal{Y}$ , and for general  $k$  with  $c(k + 1) \geq N$ . In conclusion, we have the following theorem

**Theorem 4.3.** *Let  $\mathcal{Y}$  be the generalized universal Grassmannian defined by*

$$\mathcal{Y} := \{(\Delta_1, \dots, \Delta_c, z) \in \mathrm{Gr}_{k+1}(V_1) \times \dots \times \mathrm{Gr}_{k+1}(V_c) \times \mathbb{P}^N \mid \forall i, \Delta_i([z]) = 0\}$$

*here  $V_i := H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(\delta_i))$ , and  $(k+1)c \geq N$ . Then for any strata  $\mathcal{Y}_J := (\mathbf{G} \times \mathbb{P}^J) \cap \mathcal{Y}$ , any  $\mathbf{a} \in \mathbb{Z}^c$  with  $a_i \geq \frac{\prod_{j=1}^c \delta_j^{k+1}}{\delta_i}$  for each  $i$ , we have*

$$\mathrm{Bs}(p^* \mathcal{L}(\mathbf{a}) \otimes q^* \mathcal{O}_{\mathbb{P}^N}(-1)|_{\mathcal{Y}_J}) \subset p^{-1}(G_J^\infty),$$

*where  $G_J^\infty$  is the set of points in  $\mathbf{G} := \prod_{i=1}^c \mathrm{Gr}_{k+1}(V_i)$  with positive dimension fibers in  $\mathcal{Y}_J$ .*

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